Examples in dynamical systems
Sturmian words

Pierre Arnoux

Lecture 1

Words: basic notions

- A finite **alphabet** $\mathcal{A}$ (letters $a, b, c, \ldots \in \mathcal{A}$); a **word** $W = a_0 a_1 \cdots a_{n-1}$: a finite sequence (or list) of letters.

Words: basic notions

- A finite **alphabet** $\mathcal{A}$ (letters $a, b, c, \ldots \in \mathcal{A}$); a **word** $W = a_0 a_1 \cdots a_{n-1}$: a finite sequence (or list) of letters.
- The **empty word** e (length 0); $|W| = n$ is the **length** of W.

Words: basic notions

- A finite **alphabet** $\mathcal{A}$ (letters $a, b, c, \ldots \in \mathcal{A}$); a **word** $W = a_0 a_1 \cdots a_{n-1}$: a finite sequence (or list) of letters.
- The **empty word** e (length 0); $|W| = n$ is the **length** of W.
- The number of occurences of a in W is denoted by $|W|_a$; we have $\sum_{a \in \mathcal{A}} |W|_a = |W|$.

Words: basic notions

- A finite **alphabet** $\mathcal{A}$ (letters $a, b, c, \ldots \in \mathcal{A}$); a **word** $W = a_0 a_1 \cdots a_{n-1}$: a finite sequence (or list) of letters.
- The **empty word** e (length 0); $|W| = n$ is the **length** of W.
- The number of occurences of a in W is denoted by $|W|_a$; we have $\sum_{a \in \mathcal{A}} |W|_a = |W|$.
- A **prefix** $P = a_1 \cdots a_k$; a **suffix** $S = a_{k+1} \cdots a_n$ (so $W = PS$).

Words: basic notions

- A finite **alphabet** $\mathcal{A}$ (letters $a, b, c, \ldots \in \mathcal{A}$); a **word** $W = a_0 a_1 \cdots a_{n-1}$: a finite sequence (or list) of letters.
- The **empty word** e (length 0); $|W| = n$ is the **length** of W.
- The number of occurences of a in W is denoted by $|W|_a$; we have $\sum_{a \in \mathcal{A}} |W|_a = |W|$.
- A **prefix** $P = a_1 \cdots a_k$; a **suffix** $S = a_{k+1} \cdots a_n$ (so $W = PS$).
- A **factor** $F = a_i \cdots a_j$: any contiguous block anywhere inside W.

The free monoid

- **Definition**: $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ — the set of *all* finite words over $\mathcal{A}$ (including $e \in \mathcal{A}^0$).

The free monoid

- **Definition**: $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ — the set of *all* finite words over $\mathcal{A}$ (including $e \in \mathcal{A}^0$).
- **Concatenation**: UV = write U then V. Associative, with identity e: $\mathcal{A}^*$ is a **monoid** — the **free monoid** on $\mathcal{A}$.

The free monoid

- **Definition**: $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ — the set of *all* finite words over $\mathcal{A}$ (including $e \in \mathcal{A}^0$).
- **Concatenation**: $UV =$ write U then V. Associative, with identity e: $\mathcal{A}^*$ is a **monoid** — the **free monoid** on $\mathcal{A}$.
- "Free": no relation holds except what associativity forces — exactly the words themselves, with no simplification.

The free monoid

- **Definition**: $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$ — the set of *all* finite words over $\mathcal{A}$ (including $e \in \mathcal{A}^0$).
- **Concatenation**: $UV =$ write U then V. Associative, with identity e: $\mathcal{A}^*$ is a **monoid** — the **free monoid** on $\mathcal{A}$.
- "Free": no relation holds except what associativity forces — exactly the words themselves, with no simplification.
- **Powers**: $W^n = WW \cdots W$ (n times), $W^0 = e$.

Morphisms of monoids, and substitutions

- A **monoid morphism** $\sigma : \mathcal{A}^* \to B^*$: determined freely by the images $\sigma(a)$, $a \in \mathcal{A}$, extended by $\sigma(UV) = \sigma(U)\sigma(V)$.

Morphisms of monoids, and substitutions

- A **monoid morphism** $\sigma : \mathcal{A}^* \to B^*$: determined freely by the images $\sigma(a)$, $a \in \mathcal{A}$, extended by $\sigma(UV) = \sigma(U)\sigma(V)$.
- Called a **substitution** when *non-erasing*: $\sigma(a) \neq e$ for every letter a (no letter disappears).

Morphisms of monoids, and substitutions

- A **monoid morphism** $\sigma : \mathcal{A}^* \to B^*$: determined freely by the images $\sigma(a)$, $a \in \mathcal{A}$, extended by $\sigma(UV) = \sigma(U)\sigma(V)$.
- Called a **substitution** when *non-erasing*: $\sigma(a) \neq e$ for every letter a (no letter disappears).
- **Example** (Thue–Morse): $\sigma(0) = 01$, $\sigma(1) = 10$.

Morphisms of monoids, and substitutions

- A **monoid morphism** $\sigma : \mathcal{A}^* \to B^*$: determined freely by the images $\sigma(a)$, $a \in \mathcal{A}$, extended by $\sigma(UV) = \sigma(U)\sigma(V)$.
- Called a **substitution** when *non-erasing*: $\sigma(a) \neq e$ for every letter a (no letter disappears).
- **Example** (Thue–Morse): $\sigma(0) = 01$, $\sigma(1) = 10$.
- **Example** (Fibonacci): $\sigma(0) = 01$, $\sigma(1) = 0$ — iterating from 0: $0, 01, 010, 01001, \ldots$

Languages: basic definitions

- A **language**: any subset $L \subseteq \mathcal{A}^*$.

Languages: basic definitions

- A **language**: any subset $L \subseteq \mathcal{A}^*$.
- **Finite languages**: e.g. $L = \{ab, ba, e\}$ — trivially well understood, but not expressive enough on their own.

Languages: basic definitions

- A **language**: any subset $L \subseteq \mathcal{A}^*$.
- **Finite languages**: e.g. $L = \{ab, ba, e\}$ — trivially well understood, but not expressive enough on their own.
- Operations: union $L_1 \cup L_2$; concatenation $L_1 L_2 = \{UV : U \in L_1, V \in L_2\}$; **Kleene star** L^* (the same star operation as before).

Languages: basic definitions

- A **language**: any subset $L \subseteq \mathcal{A}^*$.
- **Finite languages**: e.g. $L = \{ab, ba, e\}$ — trivially well understood, but not expressive enough on their own.
- Operations: union $L_1 \cup L_2$; concatenation $L_1 L_2 = \{UV : U \in L_1, V \in L_2\}$; **Kleene star** L^* (the same star operation as before).
- **Example**: $\{0,1\}^*$ is all binary words; $(01)^*$ is all words of the form $0101 \cdots 01$.

Factorial languages

- **Definition**: the language L: is *factorial* if, for any W in L, all the factors of W belong to L.

Factorial languages

- **Definition**: the language L: is *factorial* if, for any W in L, all the factors of W belong to L.
- **Definition**: the language L: is *right extendable* (resp. *left-extendable*) if, for any W in L, there is a letter $a \in \mathcal{A}$ such that Wa belongs to L (resp. aW belongs to L).

Factorial languages

- **Definition**: the language L: is *factorial* if, for any W in L, all the factors of W belong to L.
- **Definition**: the language L: is *right extendable* (resp. *left-extendable*) if, for any W in L, there is a letter $a \in \mathcal{A}$ such that Wa belongs to L (resp. aW belongs to L).
- **From now on, we suppose that all the languages we consider are factorial and extendable**

Complexity of a language

- **Definition**: the **complexity** of L: $p_L(n) = |L \cap \mathcal{A}^n|$, the number of length-$n$ words in L.

Complexity of a language

- **Definition**: the **complexity** of L: $p_L(n) = |L \cap \mathcal{A}^n|$, the number of length-$n$ words in L.
- If $|\mathcal{A}| = d$ we have $p(n) \leq d^n$

Complexity of a language

- **Definition**: the **complexity** of L: $p_L(n) = |L \cap \mathcal{A}^n|$, the number of length-$n$ words in L.
- If $|\mathcal{A}| = d$ we have $p(n) \leq d^n$
- The complexity is increasing : $p(n+1) \geq p(n)$.

Complexity of a language

- **Definition**: the **complexity** of L: $p_L(n) = |L \cap \mathcal{A}^n|$, the number of length-$n$ words in L.
- If $|\mathcal{A}| = d$ we have $p(n) \leq d^n$
- The complexity is increasing : $p(n+1) \geq p(n)$.
- p is *submultiplicative* ; the is, $p(m+n) \leq p(m)p(n)$.

Complexity of a language

- **Definition**: the **complexity** of L: $p_L(n) = |L \cap \mathcal{A}^n|$, the number of length-$n$ words in L.
- If $|\mathcal{A}| = d$ we have $p(n) \leq d^n$
- The complexity is increasing : $p(n+1) \geq p(n)$.
- p is *submultiplicative* ; the is, $p(m+n) \leq p(m)p(n)$.
- Hence $\lim_{n\to\infty} \frac{\log p(n)}{n}$ exists (entropy of the language)

Coding a dynamical system

- A finite **partition** of X indexed by $\mathcal{A}$: disjoint sets $(P_a)_{a\in\mathcal{A}}$ covering X.

Coding a dynamical system

- A finite **partition** of X indexed by $\mathcal{A}$: disjoint sets $(P_a)_{a\in\mathcal{A}}$ covering X.
- The **coding map** $\nu : X \to \mathcal{A}$: $\nu(x) = a$ where $x \in P_a$ — "which piece of the partition contains x".

Coding a dynamical system

- A finite **partition** of X indexed by $\mathcal{A}$: disjoint sets $(P_a)_{a\in\mathcal{A}}$ covering X.
- The **coding map** $\nu : X \to \mathcal{A}$: $\nu(x) = a$ where $x \in P_a$ — "which piece of the partition contains x".
- Given $T : X \to X$: the **itinerary** of x:

$$I(x) = \left(\nu(T^n x)\right)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$$

Coding a dynamical system

- A finite **partition** of X indexed by $\mathcal{A}$: disjoint sets $(P_a)_{a\in\mathcal{A}}$ covering X.
- The **coding map** $\nu : X \to \mathcal{A}$: $\nu(x) = a$ where $x \in P_a$ — "which piece of the partition contains x".
- Given $T : X \to X$: the **itinerary** of x:

$$I(x) = \left(\nu(T^n x)\right)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$$

- $I(x)$ records, step by step, which piece of the partition the orbit of x visits — turning a dynamical trajectory into an infinite word.

Coding a dynamical system

- A finite **partition** of X indexed by $\mathcal{A}$: disjoint sets $(P_a)_{a\in\mathcal{A}}$ covering X.
- The **coding map** $\nu : X \to \mathcal{A}$: $\nu(x) = a$ where $x \in P_a$ — "which piece of the partition contains x".
- Given $T : X \to X$: the **itinerary** of x:

$$I(x) = \left(\nu(T^n x)\right)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$$

- $I(x)$ records, step by step, which piece of the partition the orbit of x visits — turning a dynamical trajectory into an infinite word.
- We do that all the time : decimal expansion.

Infinite words and their language

- An **infinite word**: an element $x = (x_n)_{n\in\mathbb{N}}$ of $\mathcal{A}^{\mathbb{N}}$.

Infinite words and their language

- An **infinite word**: an element $x = (x_n)_{n\in\mathbb{N}}$ of $\mathcal{A}^{\mathbb{N}}$.
- The **language** of x: $L(x) = \{\text{finite factors occurring somewhere in } x\}$.

Infinite words and their language

- An **infinite word**: an element $x = (x_n)_{n\in\mathbb{N}}$ of $\mathcal{A}^{\mathbb{N}}$.
- The **language** of x: $L(x) = \{\text{finite factors occurring somewhere in } x\}$.
- **Example**: the language of the Thue–Morse word contains $0, 1, 00, 01, 10, 11, \ldots$ but not 000 or 111 (no cube — consistent with overlap-freeness).

A topology on $\mathcal{A}^{\mathbb{N}}$

- **Natural distance**: $d(x, y) = 2^{-n}$ where $n = \min\{k : x_k \neq y_k\}$ (and $d(x, x) = 0$).

A topology on $\mathcal{A}^{\mathbb{N}}$

- **Natural distance**: $d(x, y) = 2^{-n}$ where $n = \min\{k : x_k \neq y_k\}$ (and $d(x, x) = 0$).
- Two points are close exactly when they agree on a long common prefix — a compact, metrizable topology on $\mathcal{A}^{\mathbb{N}}$ (the product topology).

A topology on $\mathcal{A}^{\mathbb{N}}$

- **Natural distance**: $d(x, y) = 2^{-n}$ where $n = \min\{k : x_k \neq y_k\}$ (and $d(x, x) = 0$).
- Two points are close exactly when they agree on a long common prefix — a compact, metrizable topology on $\mathcal{A}^{\mathbb{N}}$ (the product topology).
- **Shift map** $S : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$: $S(x)_n = x_{n+1}$ (drop the first letter).

A topology on $\mathcal{A}^{\mathbb{N}}$

- **Natural distance**: $d(x, y) = 2^{-n}$ where $n = \min\{k : x_k \neq y_k\}$ (and $d(x, x) = 0$).
- Two points are close exactly when they agree on a long common prefix — a compact, metrizable topology on $\mathcal{A}^{\mathbb{N}}$ (the product topology).
- **Shift map** $S : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$: $S(x)_n = x_{n+1}$ (drop the first letter).
- S is **continuous**: if x, y agree on a long prefix, so do $S(x), S(y)$ (common prefix one letter shorter).

A topology on $\mathcal{A}^{\mathbb{N}}$

- **Natural distance**: $d(x, y) = 2^{-n}$ where $n = \min\{k : x_k \neq y_k\}$ (and $d(x, x) = 0$).
- Two points are close exactly when they agree on a long common prefix — a compact, metrizable topology on $\mathcal{A}^{\mathbb{N}}$ (the product topology).
- **Shift map** $S : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$: $S(x)_n = x_{n+1}$ (drop the first letter).
- S is **continuous**: if x, y agree on a long prefix, so do $S(x), S(y)$ (common prefix one letter shorter).
- And even Lipschitz with constant 2: $d(S(x), S(y)) \leq d(x, y)$ (exercises!).

Symbolic systems

- A **symbolic system** (subshift): a subset $X \subseteq \mathcal{A}^{\mathbb{N}}$ that is **closed** and **shift-invariant** ($S(X) \subseteq X$).

Symbolic systems

- A **symbolic system** (subshift): a subset $X \subseteq \mathcal{A}^{\mathbb{N}}$ that is **closed** and **shift-invariant** ($S(X) \subseteq X$).
- For a subshift X: $L(X) = \bigcup_{x \in X} L(x)$ is automatically factorial and extendable.

The fundamental bijection

- **Theorem**: $X \mapsto L(X)$ is a bijection between closed shift-invariant subsets of $\mathcal{A}^{\mathbb{N}}$ and factorial, extendable languages.

The fundamental bijection

- **Theorem**: $X \mapsto L(X)$ is a bijection between closed shift-invariant subsets of $\mathcal{A}^{\mathbb{N}}$ and factorial, extendable languages.
- **Inverse**: given factorial extendable L, set $X_L = \{x \in \mathcal{A}^{\mathbb{N}} : \text{every factor of } x \text{ lies in } L\}$ — closed (an intersection of clopen conditions) and shift-invariant.

The fundamental bijection

- **Theorem**: $X \mapsto L(X)$ is a bijection between closed shift-invariant subsets of $\mathcal{A}^{\mathbb{N}}$ and factorial, extendable languages.
- **Inverse**: given factorial extendable L, set $X_L = \{x \in \mathcal{A}^{\mathbb{N}} : \text{every factor of } x \text{ lies in } L\}$ — closed (an intersection of clopen conditions) and shift-invariant.
- One checks $L(X_L) = L$ (extendability lets every word of L extend to an actual point of X_L) and $X_{L(X)} = X$ (closedness of X recovers exactly the points whose factors all lie in $L(X)$).

The fundamental bijection

- **Theorem**: $X \mapsto L(X)$ is a bijection between closed shift-invariant subsets of $\mathcal{A}^{\mathbb{N}}$ and factorial, extendable languages.
- **Inverse**: given factorial extendable L, set $X_L = \{x \in \mathcal{A}^{\mathbb{N}} : \text{every factor of } x \text{ lies in } L\}$ — closed (an intersection of clopen conditions) and shift-invariant.
- One checks $L(X_L) = L$ (extendability lets every word of L extend to an actual point of X_L) and $X_{L(X)} = X$ (closedness of X recovers exactly the points whose factors all lie in $L(X)$).
- So: describing a subshift topologically, or describing it combinatorially by its language — *exactly* the same information.

The semi-conjugacy principle

- We will suppose that the partition (P_a) is **generating**, that is, the itinerary map $I : X \to \mathcal{A}^{\mathbb{N}}$ is injective : every itinerary completely characterises a point.

The semi-conjugacy principle

- We will suppose that the partition (P_a) is **generating**, that is, the itinerary map $I : X \to \mathcal{A}^{\mathbb{N}}$ is injective : every itinerary completely characterises a point.
- **Key identity**: $I(Tx)_n = \nu(T^{n+1}x) = I(x)_{n+1} = S(I(x))_n$, i.e.

$$I \circ T = S \circ I$$

The semi-conjugacy principle

- We will suppose that the partition (P_a) is **generating**, that is, the itinerary map $I : X \to \mathcal{A}^{\mathbb{N}}$ is injective : every itinerary completely characterises a point.
- **Key identity**: $I(Tx)_n = \nu(T^{n+1}x) = I(x)_{n+1} = S(I(x))_n$, i.e.

$$I \circ T = S \circ I$$

- So I intertwines T with the shift S on the subshift $X' = \overline{I(X)}$: (X, T) is (semi-)conjugate to the purely combinatorial system (X', S).

The semi-conjugacy principle

- We will suppose that the partition (P_a) is **generating**, that is, the itinerary map $I : X \to \mathcal{A}^{\mathbb{N}}$ is injective : every itinerary completely characterises a point.
- **Key identity**: $I(Tx)_n = \nu(T^{n+1}x) = I(x)_{n+1} = S(I(x))_n$, i.e.

$$I \circ T = S \circ I$$

- So I intertwines T with the shift S on the subshift $X' = \overline{I(X)}$: (X, T) is (semi-)conjugate to the purely combinatorial system (X', S).
- **This is the whole point of symbolic dynamics**: replace an analytic/topological system by a subshift, and study it combinatorially instead — via its language, complexity, factors.

Special factors

- W is **right-special** if it extends in more than one way: $RS(W) = \#\{a : Wa \in L\} - 1 \geq 1$.

Special factors

- W is **right-special** if it extends in more than one way: $RS(W) = \#\{a : Wa \in L\} - 1 \geq 1$.
- Writing $s(n) = p(n+1) - p(n)$ (the discrete derivative):

$$s(n) = \sum_{W \in L_n} RS(W)$$

Special factors

- W is **right-special** if it extends in more than one way: $RS(W) = \#\{a : Wa \in L\} - 1 \geq 1$.
- Writing $s(n) = p(n+1) - p(n)$ (the discrete derivative):

$$s(n) = \sum_{W \in L_n} RS(W)$$

- (Immediate: each $W \in L_n$ contributes $\#\{a : Wa \in L\}$ words to L_{n+1}, and summing $\#\{a : Wa \in L\} = RS(W) + 1$ over L_n gives $p(n+1) = p(n) + s(n)$.)

Special factors: the two-letter case

- On a 2-letter alphabet, $RS(W) \in \{0, 1\}$: $s(n)$ simply *counts* right-special words of length n.

Special factors: the two-letter case

- On a 2-letter alphabet, $RS(W) \in \{0, 1\}$: $s(n)$ simply *counts* right-special words of length n.
- A symmetric statement holds for **left-extendable** languages and left-special words.

Special factors: the two-letter case

- On a 2-letter alphabet, $RS(W) \in \{0, 1\}$: $s(n)$ simply *counts* right-special words of length n.
- A symmetric statement holds for **left-extendable** languages and left-special words.
- This is the tool we will use throughout: complexity growth $\Longleftrightarrow$ counting special factors.

The Coven–Hedlund lemma: statement

- **Lemma** (Coven–Hedlund, 1973): for a sequence u, the following are equivalent:

The Coven–Hedlund lemma: statement

- **Lemma** (Coven–Hedlund, 1973): for a sequence u, the following are equivalent:
- (i) u is eventually periodic;

The Coven–Hedlund lemma: statement

- **Lemma** (Coven–Hedlund, 1973): for a sequence u, the following are equivalent:
- (i) u is eventually periodic;
- (ii) the complexity $p(n)$ is bounded;

The Coven–Hedlund lemma: statement

- **Lemma** (Coven–Hedlund, 1973): for a sequence u, the following are equivalent:
- (i) u is eventually periodic;
- (ii) the complexity $p(n)$ is bounded;
- (iii) $p(n+1) = p(n)$ for *some* n.

The Coven–Hedlund lemma: proof

- **(i)⇒(ii)**: eventually periodic with preperiod N, period P: at most $N + P$ factors of any given length.

The Coven–Hedlund lemma: proof

- **(i)⇒(ii)**: eventually periodic with preperiod N, period P: at most $N + P$ factors of any given length.
- **(ii)⇒(iii)**: $p(n)$ is increasing and bounded, hence eventually constant: some n has $p(n+1) = p(n)$.

The Coven–Hedlund lemma: proof

- **(i)⇒(ii)**: eventually periodic with preperiod N, period P: at most $N + P$ factors of any given length.
- **(ii)⇒(iii)**: $p(n)$ is increasing and bounded, hence eventually constant: some n has $p(n+1) = p(n)$.
- **(iii)⇒(i)**: $p(n+1) = p(n)$ means every length-n factor extends in *exactly one* way: a well-defined map $\mathcal{A}^n \to \mathcal{A}^n$, "next factor".

The Coven–Hedlund lemma: proof

- **(i)⇒(ii)**: eventually periodic with preperiod N, period P: at most $N + P$ factors of any given length.
- **(ii)⇒(iii)**: $p(n)$ is increasing and bounded, hence eventually constant: some n has $p(n+1) = p(n)$.
- **(iii)⇒(i)**: $p(n+1) = p(n)$ means every length-n factor extends in *exactly one* way: a well-defined map $\mathcal{A}^n \to \mathcal{A}^n$, "next factor".
- Iterating from any length-n factor of u: after at most $p(n)$ steps we revisit a factor already seen — the sequence becomes periodic from there on. ■

The complexity gap

- **Consequence**: there is a genuine *gap* in possible complexity functions.

The complexity gap

- **Consequence**: there is a genuine *gap* in possible complexity functions.
- Either $p(n)$ is eventually constant (periodic case), **or** $p(n) \geq n + 1$ for *every* n — nothing in between is possible.

The complexity gap

- **Consequence**: there is a genuine *gap* in possible complexity functions.
- Either $p(n)$ is eventually constant (periodic case), **or** $p(n) \geq n+1$ for *every* n — nothing in between is possible.
- **Today's goal**: study the aperiodic sequences of *minimal* possible complexity, $p(n) = n+1$. It is not immediately clear that such sequences even **exist**!

Definition 1: Sturmian words

- **Definition**: a **Sturmian word** is an aperiodic sequence of minimal complexity, $p(n) = n + 1$.

Definition 1: Sturmian words

- **Definition**: a **Sturmian word** is an aperiodic sequence of minimal complexity, $p(n) = n + 1$.
- Since $p(1) = 2$: Sturmian words live on a 2-letter alphabet, by convention $\mathcal{A} = \{0, 1\}$.

Definition 1: Sturmian words

- **Definition**: a **Sturmian word** is an aperiodic sequence of minimal complexity, $p(n) = n + 1$.
- Since $p(1) = 2$: Sturmian words live on a 2-letter alphabet, by convention $\mathcal{A} = \{0, 1\}$.
- **Fact**: in a Sturmian word, exactly one of $00, 11$ never occurs.

Definition 1: Sturmian words

- **Definition**: a **Sturmian word** is an aperiodic sequence of minimal complexity, $p(n) = n + 1$.
- Since $p(1) = 2$: Sturmian words live on a 2-letter alphabet, by convention $\mathcal{A} = \{0, 1\}$.
- **Fact**: in a Sturmian word, exactly one of $00, 11$ never occurs.
- If 00 (resp. 11) does not occur, we say that the word is of type 1 (resp. 0).

Definition 2: balanced words

- **Definition**: $u \in \{0,1\}^{\mathbb{N}}$ is **balanced** if, for *any* two factors U, V of the *same* length:

$$\big||U|_1 - |V|_1\big| \leq 1$$

Definition 2: balanced words

- **Definition**: $u \in \{0,1\}^{\mathbb{N}}$ is **balanced** if, for *any* two factors U, V of the *same* length:

$$\big||U|_1 - |V|_1\big| \leq 1$$

- A window of a given size contains at most two different numbers of 1's.

Definition 3: rotation sequences

- **Definition**: u is a **rotation sequence** if there are α, β with, for every n, either

$$u_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$$

or

$$u_n = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil$$

Definition 3: rotation sequences

- **Definition**: u is a **rotation sequence** if there are α, β with, for every n, either

$$u_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$$

or $$u_n = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil$$

- Equivalently : the itinerary of the point β under rotation by α, coded by a 2-interval partition of the circle.

Definition 3: rotation sequences

- **Definition**: u is a **rotation sequence** if there are α, β with, for every n, either

$$u_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$$

or $$u_n = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil$$

- Equivalently : the itinerary of the point β under rotation by α, coded by a 2-interval partition of the circle.
- Many disguises: a billiard sequence in a corner, a line through a grid, a mechanical/Christoffel word, a 1D quasicrystal.

Sturmian $\Longleftrightarrow$ aperiodic and balanced

- **Theorem** (Morse–Hedlund, 1940): an infinite word is Sturmian if and only if it is aperiodic and balanced.

Sturmian $\Longleftrightarrow$ aperiodic and balanced

- **Theorem** (Morse–Hedlund, 1940): an infinite word is Sturmian if and only if it is aperiodic and balanced.
- This is *the* founding theorem of the subject: it turns a dynamical/combinatorial condition (minimal complexity) into a purely arithmetic one (balance).

Sturmian $\iff$ aperiodic and balanced

- **Theorem** (Morse–Hedlund, 1940): an infinite word is Sturmian if and only if it is aperiodic and balanced.
- This is *the* founding theorem of the subject: it turns a dynamical/combinatorial condition (minimal complexity) into a purely arithmetic one (balance).
- We prove both directions — this is the main result of today's lecture.

Direction 1: Not Sturmian $\Rightarrow$ not balanced

- Suppose u is aperiodic but *not* Sturmian: by Coven–Hedlund, $p(n) \geq n+2$ for some n. Take the *first* such n.

Direction 1: Not Sturmian $\Rightarrow$ not balanced

- Suppose u is aperiodic but *not* Sturmian: by Coven–Hedlund, $p(n) \geq n+2$ for some n. Take the *first* such n.
- Small cases ($n \leq 2$) are checked directly by inspection; assume $n > 2$ from now on.

Direction 1: Not Sturmian $\Rightarrow$ not balanced

- Suppose u is aperiodic but *not* Sturmian: by Coven–Hedlund, $p(n) \geq n+2$ for some n. Take the *first* such n.
- Small cases ($n \leq 2$) are checked directly by inspection; assume $n > 2$ from now on.
- **Lemma**: if W is right-special, every **suffix** of W is right-special too. (Symmetrically: every prefix of a left-special word is left-special.)

Direction 1: Not Sturmian $\Rightarrow$ not balanced

- Suppose u is aperiodic but *not* Sturmian: by Coven–Hedlund, $p(n) \geq n+2$ for some n. Take the *first* such n.
- Small cases ($n \leq 2$) are checked directly by inspection; assume $n > 2$ from now on.
- **Lemma**: if W is right-special, every **suffix** of W is right-special too. (Symmetrically: every prefix of a left-special word is left-special.)
- **Proof**: if $Wa, Wb \in L$ ($a \neq b$) and $W = CW'$, then $W'a, W'b$ are factors of $Wa, Wb \in L$ (factoriality): W' is right-special too.

Direction 1: identifying U, V

- $p(n-1) = n$ exactly (first jump, so no jump yet at $n-1$): by the suffix lemma, the (at least two) right-special words of length $n-1$ all have right-special *suffixes* of every shorter length.

Direction 1: identifying U, V

- $p(n-1) = n$ exactly (first jump, so no jump yet at $n-1$): by the suffix lemma, the (at least two) right-special words of length $n-1$ all have right-special *suffixes* of every shorter length.
- But for lengths $< n-1$ there is only *one* right-special word each (no jump yet): so all these right-special words of length $n-1$ share the *same* suffix of length $n-2$, call it W.

Direction 1: identifying U, V

- $p(n-1) = n$ exactly (first jump, so no jump yet at $n-1$): by the suffix lemma, the (at least two) right-special words of length $n-1$ all have right-special *suffixes* of every shorter length.
- But for lengths $< n-1$ there is only *one* right-special word each (no jump yet): so all these right-special words of length $n-1$ share the *same* suffix of length $n-2$, call it W.
- They can only differ in their *first* letter: the two right-special words of length $n-1$ are exactly

$$U = 0W, \qquad V = 1W$$

Direction 1: concluding

- $U = 0W$ right-special: $0W0, 0W1 \in L$. $V = 1W$ right-special: $1W0, 1W1 \in L$.

Direction 1: concluding

- $U = 0W$ right-special: $0W0, 0W1 \in L$. $V = 1W$ right-special: $1W0, 1W1 \in L$.
- In particular, $0W0$ and $1W1$ both occur — **two factors of the same length**, with

$$|0W0|_1 = |W|_1, \qquad |1W1|_1 = |W|_1 + 2$$

Direction 1: concluding

- $U = 0W$ right-special: $0W0, 0W1 \in L$. $V = 1W$ right-special: $1W0, 1W1 \in L$.
- In particular, $0W0$ and $1W1$ both occur — **two factors of the same length**, with

$$|0W0|_1 = |W|_1, \qquad |1W1|_1 = |W|_1 + 2$$

- A difference of 2: u is **not balanced**. ■

Direction 1: concluding

- $U = 0W$ right-special: $0W0, 0W1 \in L$. $V = 1W$ right-special: $1W0, 1W1 \in L$.
- In particular, $0W0$ and $1W1$ both occur — **two factors of the same length**, with

$$|0W0|_1 = |W|_1, \qquad |1W1|_1 = |W|_1 + 2$$

- A difference of 2: u is **not balanced**. ■
- **Contrapositive**: aperiodic + balanced $\Rightarrow$ no complexity jump anywhere $\Rightarrow$ (Coven–Hedlund) $p(n) = n + 1$ always: **Sturmian**. One direction done.

Direction 2: Sturmian $\Rightarrow$ balanced — strategy

- The converse needs more care. Suppose, for contradiction, u is Sturmian but **not** balanced.

Direction 2: Sturmian $\Rightarrow$ balanced — strategy

- The converse needs more care. Suppose, for contradiction, u is Sturmian but **not** balanced.
- One shows: not balanced $\Rightarrow$ there is a *minimal* word W with both $0W0$ and $1W1$ occurring in u (take a shortest pair of same-length factors with 1-counts differing by 2, and pass to their common "core").

Direction 2: Sturmian $\Rightarrow$ balanced — strategy

- The converse needs more care. Suppose, for contradiction, u is Sturmian but **not** balanced.
- One shows: not balanced $\Rightarrow$ there is a *minimal* word W with both $0W0$ and $1W1$ occurring in u (take a shortest pair of same-length factors with 1-counts differing by 2, and pass to their common "core").
- W is nonempty (else $00, 11$ both occur, contradicting the Fact of Definition 1).

Direction 2: Sturmian $\Rightarrow$ balanced — strategy

- The converse needs more care. Suppose, for contradiction, u is Sturmian but **not** balanced.
- One shows: not balanced $\Rightarrow$ there is a *minimal* word W with both $0W0$ and $1W1$ occurring in u (take a shortest pair of same-length factors with 1-counts differing by 2, and pass to their common "core").
- W is nonempty (else $00, 11$ both occur, contradicting the Fact of Definition 1).
- This *minimal* W must be a **palindrome**.

Palindromes

- **Definition**: a word is a **palindrome** if it reads the same forwards and backwards: $W = w_1 w_2 \cdots w_k$ is a palindrome iff $w_i = w_{k+1-i}$ for all i.

Palindromes

- **Definition**: a word is a **palindrome** if it reads the same forwards and backwards: $W = w_1 w_2 \cdots w_k$ is a palindrome iff $w_i = w_{k+1-i}$ for all i.
- Everyday examples: `kayak`, `level`, `racecar`, `deified` — and, on a 2-letter alphabet, words like 010, 0110, 10101.

Palindromes

- **Definition**: a word is a **palindrome** if it reads the same forwards and backwards: $W = w_1 w_2 \cdots w_k$ is a palindrome iff $w_i = w_{k+1-i}$ for all i.
- Everyday examples: `kayak`, `level`, `racecar`, `deified` — and, on a 2-letter alphabet, words like 010, 0110, 10101.
- a better one : A man, a plan, a canal : Panama!

Palindromes

- **Definition**: a word is a **palindrome** if it reads the same forwards and backwards: $W = w_1 w_2 \cdots w_k$ is a palindrome iff $w_i = w_{k+1-i}$ for all i.
- Everyday examples: `kayak`, `level`, `racecar`, `deified` — and, on a 2-letter alphabet, words like 010, 0110, 10101.
- a better one : A man, a plan, a canal : Panama!
- Fact (exercise): the minimal W from the previous frame is always of this symmetric form.

W is bispecial

- $0W0, 1W1 \in L$ directly gives: $0W, 1W \in L$ (W is **left-special**), and $W0, W1 \in L$ (W is **right-special**).

W is bispecial

- $0W0, 1W1 \in L$ directly gives: $0W, 1W \in L$ (W is **left-special**), and $W0, W1 \in L$ (W is **right-special**).
- Sturmian: $s(n) = 1$ for every n, on a 2-letter alphabet — exactly *one* right-special word of each length, and (symmetrically) exactly *one* left-special word of each length.

W is bispecial

- $0W0, 1W1 \in L$ directly gives: $0W, 1W \in L$ (W is **left-special**), and $W0, W1 \in L$ (W is **right-special**).
- Sturmian: $s(n) = 1$ for every n, on a 2-letter alphabet — exactly *one* right-special word of each length, and (symmetrically) exactly *one* left-special word of each length.
- So W is **the** unique right-special *and* the unique left-special word of its length: a **bispecial factor**.

Not all four extensions can occur

- **Claim**: among $0W0, 0W1, 1W0, 1W1$, *not all four* occur in u.

Not all four extensions can occur

- **Claim**: among $0W0, 0W1, 1W0, 1W1$, *not all four* occur in u.
- **Proof**: suppose all four did. Then $0W0, 1W0 \in L$ makes $W0$ left-special; and $0W1, 1W1 \in L$ makes $W1$ left-special.

Not all four extensions can occur

- **Claim**: among $0W0, 0W1, 1W0, 1W1$, *not all four* occur in u.
- **Proof**: suppose all four did. Then $0W0, 1W0 \in L$ makes $W0$ left-special; and $0W1, 1W1 \in L$ makes $W1$ left-special.
- Two *different* words ($W0 \neq W1$) of the same length, both left-special — contradicting the uniqueness from the previous frame.

Not all four extensions can occur

- **Claim**: among $0W0, 0W1, 1W0, 1W1$, *not all four* occur in u.
- **Proof**: suppose all four did. Then $0W0, 1W0 \in L$ makes $W0$ left-special; and $0W1, 1W1 \in L$ makes $W1$ left-special.
- Two *different* words ($W0 \neq W1$) of the same length, both left-special — contradicting the uniqueness from the previous frame.
- So at least one of $0W1, 1W0$ is missing. (We already know $0W0, 1W1 \in L$.)

Why W is a palindrome

- W is the *unique* bispecial factor of its length, sitting between a matched pair $0W0, 1W1$, with (by the previous frame) an asymmetric "missing corner" among $0W1, 1W0$.

Why W is a palindrome

- W is the *unique* bispecial factor of its length, sitting between a matched pair $0W0, 1W1$, with (by the previous frame) an asymmetric "missing corner" among $0W1, 1W0$.
- One shows — comparing an occurrence of $0W0$ with an occurrence of $1W1$, and using recurrence together with minimality of $|W|$ — that any mismatch between W and its reversal W^R could be used to build a *strictly shorter* word with the same defining property $0W'0, 1W'1 \in L$.

Why W is a palindrome

- W is the *unique* bispecial factor of its length, sitting between a matched pair $0W0, 1W1$, with (by the previous frame) an asymmetric "missing corner" among $0W1, 1W0$.
- One shows — comparing an occurrence of $0W0$ with an occurrence of $1W1$, and using recurrence together with minimality of $|W|$ — that any mismatch between W and its reversal W^R could be used to build a *strictly shorter* word with the same defining property $0W'0, 1W'1 \in L$.
- This contradicts minimality of W unless $W = W^R$.

The overlap argument

- Since $0W0, 1W1 \in L$: both $0W$ and $1W$ occur (W is left-special), and both $W0, W1$ occur (W is right-special).

The overlap argument

- Since $0W0, 1W1 \in L$: both $0W$ and $1W$ occur (W is left-special), and both $W0, W1$ occur (W is right-special).
- Since p has *no* jumps (Sturmian): W is the *unique* special word of its length — so *not* both $0W1$ and $1W0$ can occur alongside $0W0, 1W1$. Suppose $|W| = n$.

The overlap argument

- Since $0W0, 1W1 \in L$: both $0W$ and $1W$ occur (W is left-special), and both $W0, W1$ occur (W is right-special).
- Since p has *no* jumps (Sturmian): W is the *unique* special word of its length — so *not* both $0W1$ and $1W0$ can occur alongside $0W0, 1W1$. Suppose $|W| = n$.
- Say $1W0 \notin L$. Take an occurrence of $1W1$ in u at index k, and consider a factor V of length $2n+2$ of u with $1W1$ as a prefix : it contains $n+1$ factors of length $n+1$. but $0W$ cannot occur in V: it cannot occur at index k and $k+n+1$, and it cannot occur at intermediate positions since W is a palindrome.

The overlap argument

- Since $0W0, 1W1 \in L$: both $0W$ and $1W$ occur (W is left-special), and both $W0, W1$ occur (W is right-special).
- Since p has *no* jumps (Sturmian): W is the *unique* special word of its length — so *not* both $0W1$ and $1W0$ can occur alongside $0W0, 1W1$. Suppose $|W| = n$.
- Say $1W0 \notin L$. Take an occurrence of $1W1$ in u at index k, and consider a factor V of length $2n+2$ of u with $1W1$ as a prefix : it contains $n+1$ factors of length $n+1$. but $0W$ cannot occur in V: it cannot occur at index k and $k+n+1$, and it cannot occur at intermediate positions since W is a palindrome.
- With $0W$ excluded there, V has *no* special word of length $|W|+1$: by the same counting as Coven–Hedlund, u becomes eventually periodic — contradicting aperiodicity. ■

The equivalence, complete

- Combining both directions:

$$\text{Sturmian} \iff \text{aperiodic and balanced}$$

The equivalence, complete

- Combining both directions:

$$\text{Sturmian} \iff \text{aperiodic and balanced}$$

- A two-way translation between **dynamics** (minimal complexity, a statement about a symbolic system) and **arithmetic** (balance, a statement about digit counts).

The equivalence, complete

- Combining both directions:

$$\text{Sturmian} \iff \text{aperiodic and balanced}$$

- A two-way translation between **dynamics** (minimal complexity, a statement about a symbolic system) and **arithmetic** (balance, a statement about digit counts).
- Everything that follows — rotations, substitutions, continued fractions — exploits *both* sides of this dictionary.