Examples in dynamical systems
Sturmian words

Pierre Arnoux

Lecture 2

Definition 1: Sturmian words

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- **Fact**: in a Sturmian word, exactly one of $00, 11$ never occurs.
- If 00 (resp. 11) does not occur, we say that the word is of type 1 (resp. 0).

Definition 2: balanced words

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- A window of a given size contains at most two different numbers of 1's.

Definition 3: rotation sequences

- **Definition**: u is a **rotation sequence** if there are α, β with, for every n, either

$$u_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$$

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- Many disguises: a billiard sequence in a corner, a line through a grid, a mechanical/Christoffel word, a 1D quasicrystal.

The main theorem

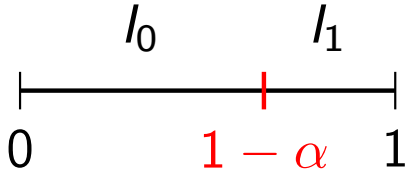
Theorem

For a binary sequence u, the following conditions are equivalent:

1. *u is a sturmian sequence*
2. *u is balanced and aperiodic*
3. *u is a rotation sequence with irrational angle α*

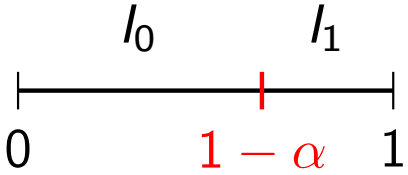
We have proved that 1. and 2. are equivalent.
We will now give two proofs that 3. implies 1. and 2.; the converse is more difficult.

Coding a rotation



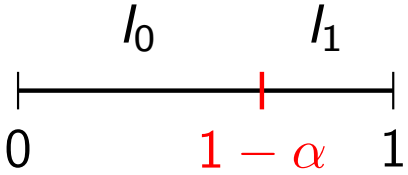
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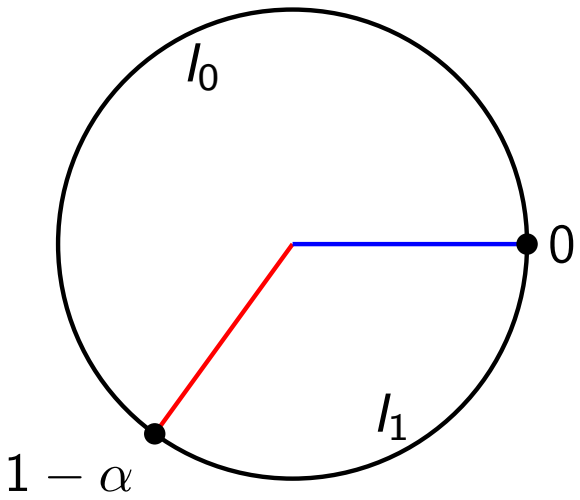
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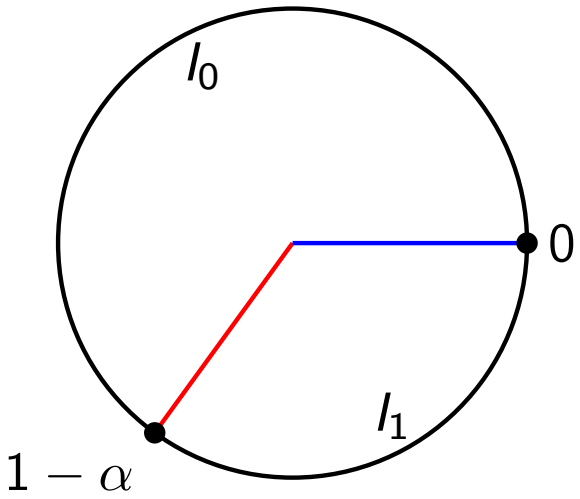
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- Code x by $\nu(x) = 0$ or 1 according to which piece x lies in.
- Itinerary of x under $R_\alpha(x) = x + \alpha \bmod 1$: $u_n = \nu(R_\alpha^n x)$.

The same partition, on the circle



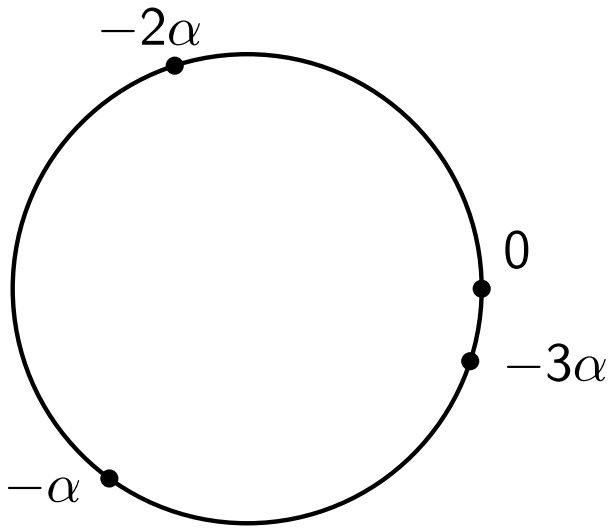
- Same partition, wrapped onto $S^1 = \mathbb{R}/\mathbb{Z}$: two arcs I_0, I_1 meeting at 0 and at $1-\alpha$.

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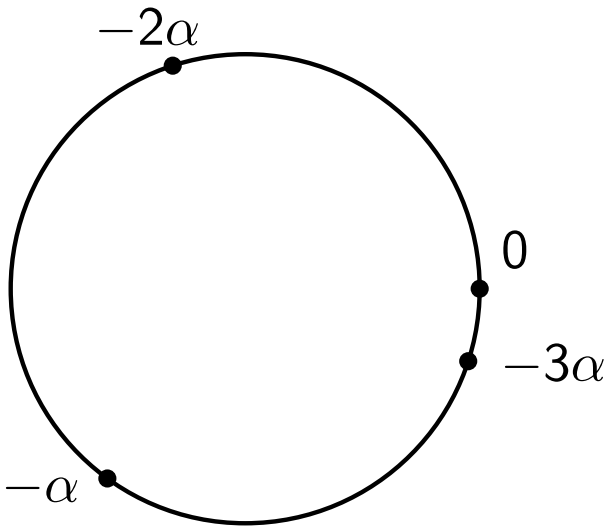
- Same partition, wrapped onto $S^1 = \mathbb{R}/\mathbb{Z}$: two arcs I_0, I_1 meeting at 0 and at $1-\alpha$.
- R_α rotates the whole circle by angle α (fraction of a turn) at each step.

Refining by the first iterates



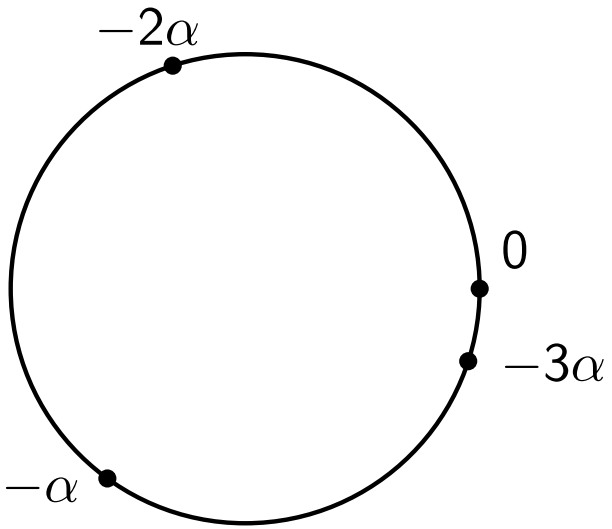
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- R_α^{-1} of this boundary: $\{-\alpha, -2\alpha\}$; R_α^{-2}: $\{-2\alpha, -3\alpha\}$; and so on.
- After the first 3 iterates: boundary points of the refined partition are exactly $\{0, -\alpha, -2\alpha, -3\alpha\} \bmod 1$ — **4 points**.

Atoms of the partition, and words

- **General pattern**: the partition into length-n itineraries has boundary exactly at $\{-j\alpha \bmod 1 : j = 0, 1, \ldots, n\}$ — $n+1$ points (all distinct, since α is irrational).

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- Each **atom** (piece) of this refined partition corresponds to exactly *one* possible length-n word: points in the same atom share the same itinerary, by construction.
- So counting atoms *is* counting factors! (It is true, though not entirely immediate, that these atoms are actual intervals — connected sets — rather than scattered unions; this uses that R_α is order-preserving.)

Counting the pieces

- $n+1$ points on a circle bound exactly $n+1$ arcs: so there are at most $n+1$ distinct length-n itineraries.

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$p(n) = n+1$ exactly: rotation sequences are Sturmian. ■

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- **Claim**: for a fixed n and *every* m, S takes only **two** values, which differ by 1. In other words, u is *balanced*.

Cancellation

- Write the sum term by term:

$$\begin{aligned} S = &\left[\lfloor (m+1)\alpha+\beta\rfloor - \lfloor m\alpha+\beta\rfloor\right] \\ &+ \left[\lfloor (m+2)\alpha+\beta\rfloor - \lfloor (m+1)\alpha+\beta\rfloor\right] \\ &+ \cdots \\ &+ \left[\lfloor (m+n)\alpha+\beta\rfloor - \lfloor (m+n-1)\alpha+\beta\rfloor\right] \end{aligned}$$

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- Put $x = m\alpha+\beta$: $S = \lfloor x+n\alpha\rfloor - \lfloor x\rfloor$.

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- More precisely, $S = \lfloor n\alpha \rfloor + 1$ exactly when $\{m\alpha + \beta\} \geq 1 - \{n\alpha\}$: the point $R_\alpha^m \beta$ lies in an interval of length $\{n\alpha\}$ (compare with the atoms of the first proof).

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- Any two factors of the same length n have numbers of 1's in $\{\lfloor n\alpha \rfloor, \lfloor n\alpha \rfloor + 1\}$: **rotation sequences are balanced**.

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- Two proofs, two properties: the first one *counts factors* (complexity $n + 1$); the second one *counts* 1*'s in windows* (balance).

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- **Another route**: the recoding of the next section reads α off the discrete continued fraction of u, digit by digit.

Exercises

Exercise 1 (recurrence). Let u be a Sturmian sequence. Show that u is **recurrent**: every factor of u occurs infinitely often (use complexity).

Exercises

Exercise 2 (frequency). Let u be a Sturmian sequence (use balance).
(a) Show that the **frequency of** 1 exists:
$\alpha = \lim_{N\to\infty} \frac{1}{N}|u_0 \cdots u_{N-1}|_1$.
(b) Show that every factor of length n contains between $n\alpha - 1$ and $n\alpha + 1$ ones.

Exercises

Exercise 3. Let u be Sturmian and let Ω be its **orbit closure**: the set of sequences all of whose factors are factors of u (equivalently, the limits of shifts of u). Show that every $w \in \Omega$ has a frequency of 1, equal to the frequency α of u.

Exercises

Exercise 4. The frequency α of a Sturmian sequence is **irrational**.

Exercises

Exercise 5 (minimality). A Sturmian sequence is **minimal**: every factor occurs with bounded gaps (equivalently, Ω has no proper closed shift-invariant subset).

Exercises

Exercise 6. Show that a balanced sequence cannot contain both 00 and 11. (This is the Fact of Definition 1.)

Exercises

Exercise 7. Let $\alpha = \frac{\sqrt{5}-1}{2}$, $\beta = 0$ and $u_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$. Compute $u_0, \ldots, u_9$. Which type is u? Check that $p(n) = n+1$ for $n \leq 4$.

Exercises

Exercise 8. If $\alpha = p/q$ is rational, show that $u_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$ is periodic. (Irrationality is really needed.)

Exercises

Exercise 9 ($\star$). Let α be irrational. Compare $u_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor$ and $u'_n = \lceil (n+1)\alpha + \beta \rceil - \lceil n\alpha + \beta \rceil$ for $n \in \mathbb{Z}$. Show that $u = u'$ unless $k\alpha + \beta \in \mathbb{Z}$ for some (unique) k, and that in that case they differ exactly at the positions $k-1, k$, with the pattern 10 for u and 01 for u'. (We will meet these two sequences again as the "fixed pair" of a Sturmian system.)

Exercises

Exercise 10 ($\star$). Show that a Sturmian sequence has exactly one *right special* factor of each length n (a factor w such that both $w0$ and $w1$ are factors), and that these right special factors are suffixes of one another.
Show the same for *left-special* factors, and deduce that, in a Sturmian system, there is a unique left special infinite word: a unique sequence u such that $0u$ and $1u$ are sturmian.

Exercises

Exercise 11 ($\star$). Can we define 2-sided sturmian sequences and systems?

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- **Type** 0: 11 never occurs (every 1 is isolated, followed by a 0).
- **Type** 1: 00 never occurs (every 0 is isolated).
- The symmetry $0 \leftrightarrow 1$ exchanges the two types: it is enough to study type 0.

Two substitutions

- A **substitution**: a monoid morphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ sending letters to nonempty words — extends to (bi-)infinite words letter-wise, $\sigma(u \cdot v) = \sigma(u) \cdot \sigma(v)$.

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- $$\sigma_0 : 0 \mapsto 0,\ 1 \mapsto 10 \qquad \sigma_1 : 0 \mapsto 01,\ 1 \mapsto 1$$
- We can associate a matrix (abelianisation) to each substitution

Recoding a type-0 sequence

- **Lemma**: for u bi-infinite Sturmian of type 0 with $u_{-1} = 0$: there is a *unique* v with $u = \sigma_0(v)$. (If $u_{-1} = 1$: unique v with $u = S\sigma_0(v)$.)

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- **Proof**: type 0 means every 1 is followed by 0 (no "11"): u decomposes *uniquely* into blocks "10" and "0" — reading off which block occurred at each position defines v exactly.
- We say u **recodes** to v under σ_0.

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- **Proof**: unbalanced v has $a0W0$ and $1W1$ occurring (some letter a). Then $\sigma_0(a0W0) = \cdots 00\,\sigma_0(W)\,0$ and $\sigma_0(1W1) = 10\,\sigma_0(W)\,10$: the images contain $00\,\sigma_0(W)\,0$ and $10\,\sigma_0(W)\,1$, of equal length, 1-counts differing by 2.

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- **Corollary**: if u (type 0) recodes to v, then v is again **Sturmian** — balance transfers (contrapositive of the lemma), and eventual periodicity transfers too (image of a periodic sequence under a substitution is periodic).

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- So recoding sends canonical pairs to canonical pairs: we can **iterate**.

A discrete continued fraction: the adic expansion

- **Theorem**: for any fixed pair u, u' of type 0, there is a *unique* sequence of positive integers (a_n) and pairs $(v^{(n)}, v'^{(n)})$ with

$$u = \sigma_0^{a_0}\sigma_1^{a_1}\sigma_0^{a_2}\cdots\sigma_0^{a_{2n}}\left(v^{(n)}\right)$$

$$u' = \sigma_0^{a_0}\sigma_1^{a_1}\sigma_0^{a_2}\cdots\sigma_0^{a_{2n}}\left(v'^{(n)}\right)$$

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A discrete continued fraction: the adic expansion

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- **Idea**: recode repeatedly; each recoded system is again type 0 or type 1, and we apply σ_0 (resp. σ_1) for as long as the type stays the same — exactly a_i times before the type switches.
- The sequence (a_n) **completely determines** the pair u, u' — a genuine positional numeral system for Sturmian sequences.

The Rauzy rules

- This gives an effective algorithm. Build words U_n, U'_n: $U_0 = 0$, $U'_0 = 1$, and at each step, **either**

$$U_{n+1} = U_n,\ U'_{n+1} = U'_n U_n$$

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- **Euclid's algorithm, transposed to words**: the lengths $(|U_n|, |U'_n|)$ evolve by $(\ell, \ell') \mapsto (\ell, \ell + \ell')$ or $(\ell + \ell', \ell')$, i.e. Euclid's algorithm on integers run backwards, and the a_n are its successive quotients. Everything here is discrete: no real number has appeared yet.

The associated sequence of matrices

- **Abelianisation**: to a substitution σ we associate $M_\sigma = \left(|\sigma(j)|_i\right)_{i,j}$, and $M_{\sigma\tau} = M_\sigma M_\tau$. Here

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

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- These matrices have non-negative integer entries and determinant 1. Conversely, every matrix of $SL(2,\mathbb{N})$ is a unique product of M_0 and M_1: that is, the monoid $SL(2,\mathbb{N})$ is free on M_0, M_1 — Euclid's algorithm again.

Recoding, and the rotation angle

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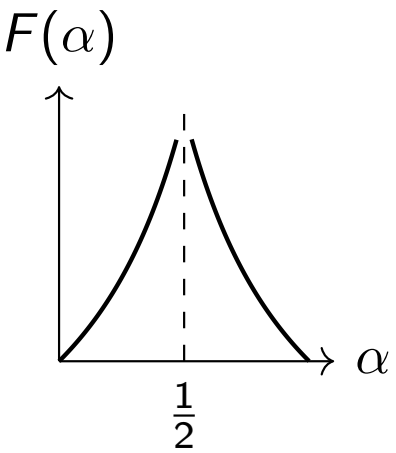
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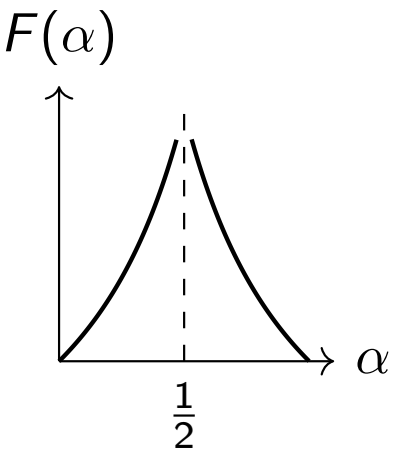
- Recoding by σ_0 is exactly rotation by $\alpha \mapsto \dfrac{\alpha}{1 - \alpha}$: an explicit, elementary formula for the new angle.

The Farey (additive) continued fraction



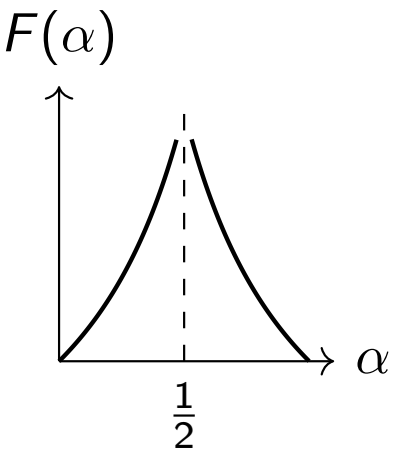
- $F(\alpha) = \dfrac{\alpha}{1-\alpha}$ for $\alpha < \frac{1}{2}$ (type 0, via σ_0). For $\alpha > \frac{1}{2}$ (type 1, via σ_1) recoding gives $\frac{2\alpha-1}{\alpha}$; exchanging the letters $0 \leftrightarrow 1$ ($\alpha \mapsto 1-\alpha$) gives the mirror branch $\dfrac{1-\alpha}{\alpha}$.

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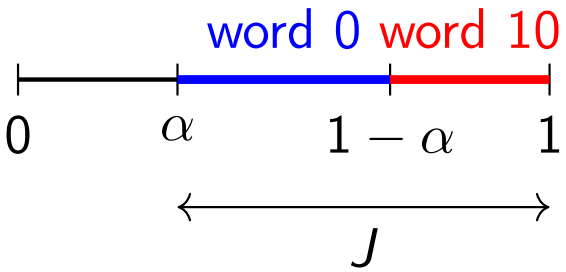


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- The **Farey map**: the "slow" (additive) continued fraction algorithm.
- Counting the consecutive recoding steps of the same type gives the exponents a_n of the adic expansion. They are the partial quotients of $\dfrac{\alpha}{1-\alpha}$, that is

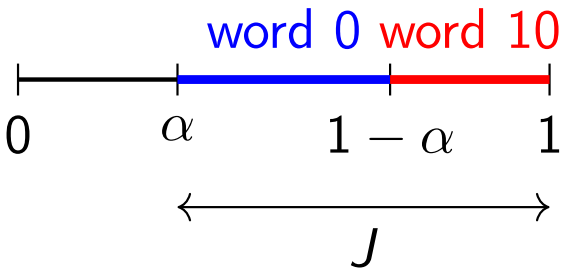
$$\alpha = [0;\, a_0 + 1,\, a_1,\, a_2, \ldots].$$

Geometric interpretation: induction on an interval

- Type 0 ($\alpha < \frac{1}{2}$). Let $J = [\alpha, 1) = R_\alpha(I_0)$: the points whose previous letter is 0.

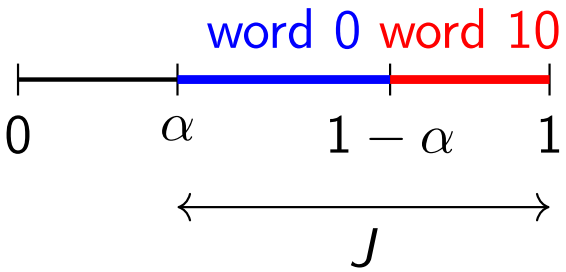


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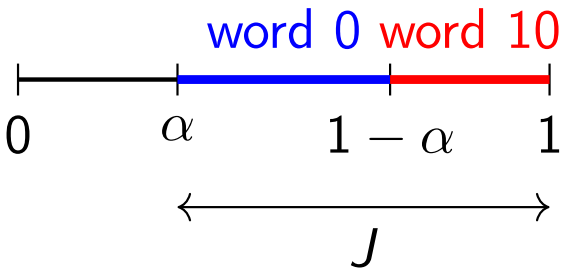
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- Rescale J to length 1: rotation by $\alpha' = \dfrac{\alpha}{1-\alpha}$, and its coding is the recoded sequence v. Iterating: the continued fraction algorithm.

The usual continued fraction

- The **Gauss map** $G(\alpha) = \{1/\alpha\} = \frac{1}{\alpha} - \left\lfloor \frac{1}{\alpha} \right\rfloor$ on $(0,1)$ gives the usual expansion $\alpha = [0; c_1, c_2, \ldots]$, with $c_n = \left\lfloor 1/G^{n-1}(\alpha) \right\rfloor$.

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- **Geometrically**: inducing R_α on $[0, \alpha)$ and rescaling gives a rotation by $-G(\alpha)$, conjugate to $R_{G(\alpha)}$ by a reflection.