Examples in dynamical systems hyperbolic geometry and the modular surface

Pierre Arnoux

Lecture 7

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- We will use a trick: we compute *one* length, and obtain everything else by symmetry.

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- **Proof, minimality**: let $\delta(t) = x(t) + iy(t)$, $0 \leq t \leq b$, with $y(0) = 1$ and $y(b) = e^a$. Then

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- This is the only length we compute: for everything else we use the action of a group.

The unimodular group acts on the hyperbolic plane

- **Proposition**: the group $\mathrm{SL}(2,\mathbb{R})$ of real matrices of determinant 1 acts on $\mathbb{H}$ by

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- The matrix $-\mathrm{Id}$ acts trivially: the effective group is $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm\mathrm{Id}\}$.

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- **Proof**: left as an exercise.

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- But the height $\Im z$ is multiplied by the same factor, so the *hyperbolic* length of the tangent vector does not change.
- **Corollary**: the action of $\mathrm{SL}(2,\mathbb{R})$ preserves the length of curves, the distance between points, and the geodesics.

The cross ratio

- **Definition**: the cross ratio of four points is

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- **Proof**: the argument of $\frac{z_1 - z_3}{z_1 - z_4}$ is the angle $\widehat{z_4 z_1 z_3}$. If the cross ratio is real, the angles $\widehat{z_4 z_1 z_3}$ and $\widehat{z_4 z_2 z_3}$ are equal mod π; by elementary geometry, z_2 is on the circle through z_4, z_1, z_3.

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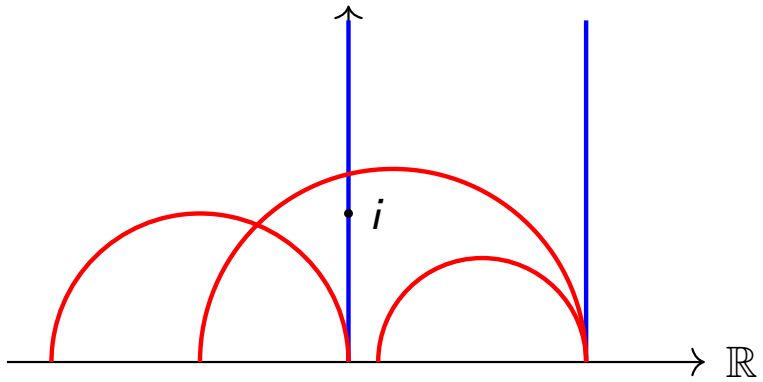
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- The action also preserves $\mathbb{R}$, and, being conformal, it preserves orthogonality. So it acts on the family of circles orthogonal to $\mathbb{R}$.

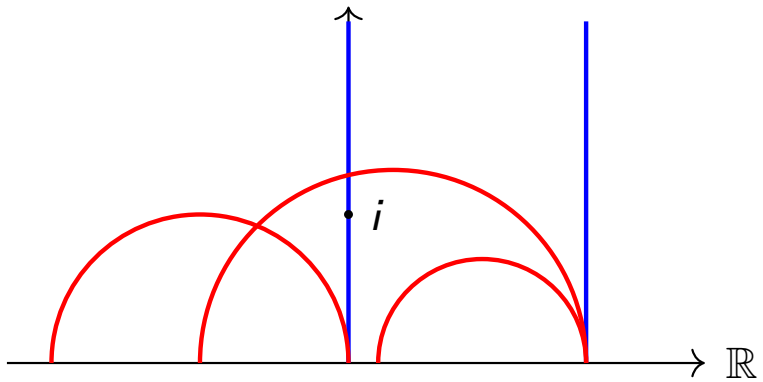
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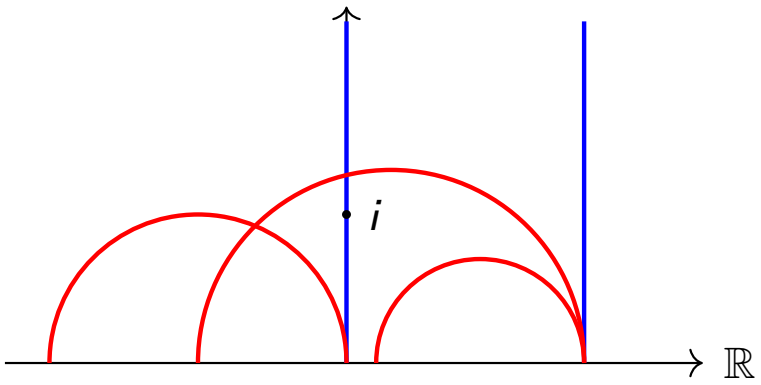
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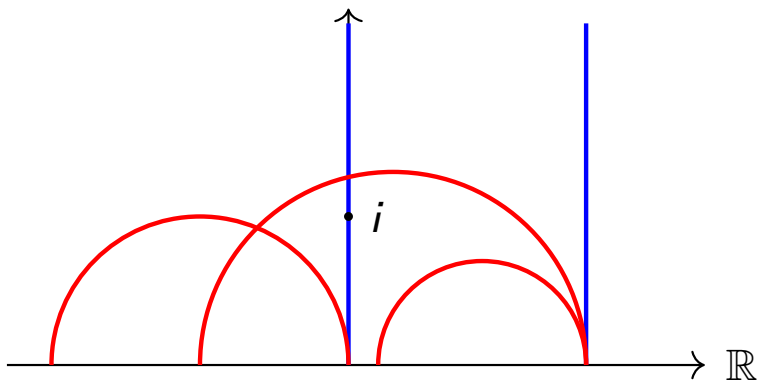
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- Through any tangent direction there is exactly one such curve: there are no other geodesics.



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- All these subgroups, *acting on the right*, define flows on $T^1\mathbb{H} = \mathrm{PSL}(2,\mathbb{R})$: $x \mapsto x\,g_t$, $x \mapsto x\,h_s^\pm$, ...

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- Hence the name **geodesic flow**.

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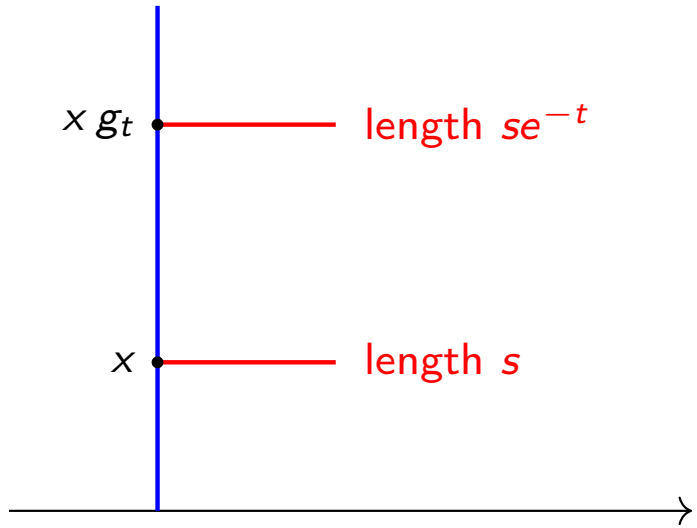
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- $h_s^- \cdot i = \dfrac{i}{1+si} = \dfrac{s+i}{1+s^2}$: a circle tangent to $\mathbb{R}$ at 0. The negative horocycles are based at 0 (the positive ones are based at ∞).

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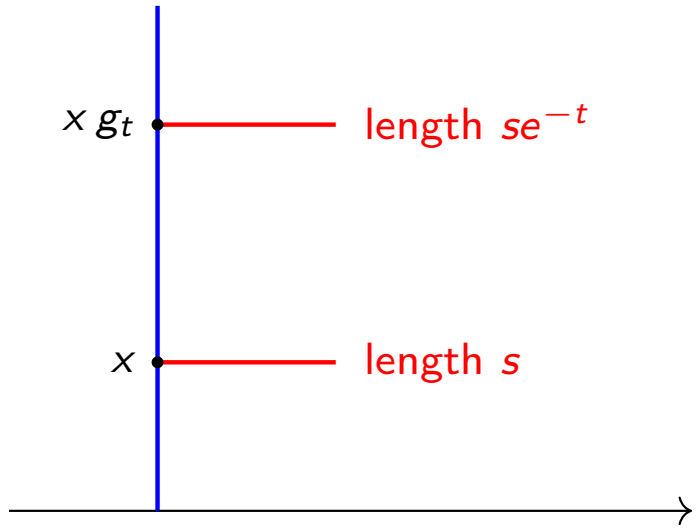


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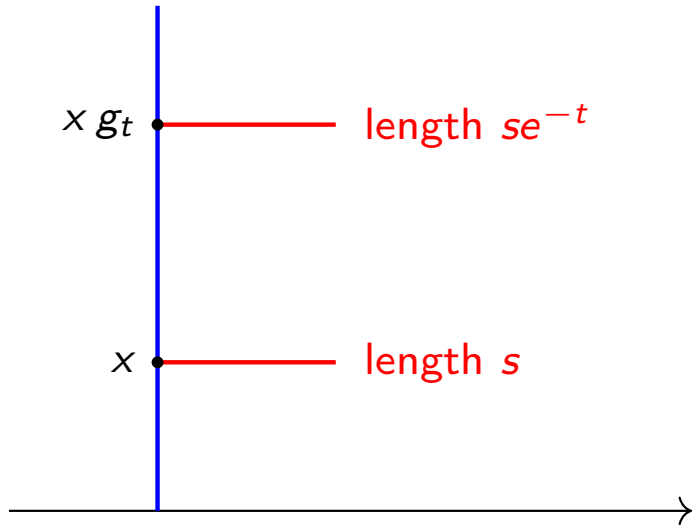


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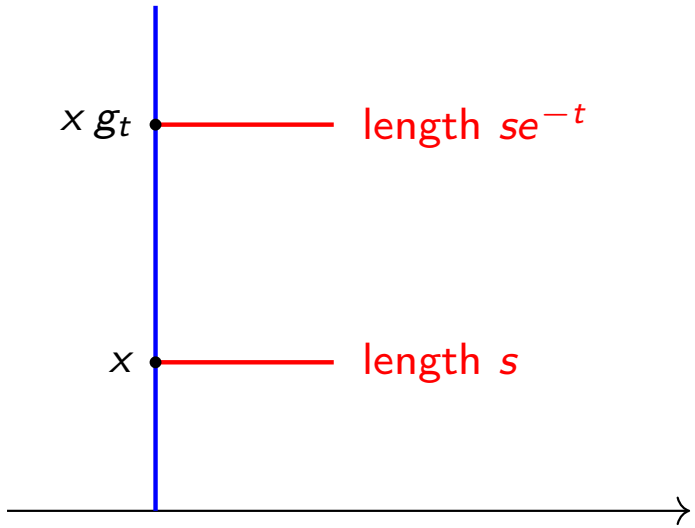


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- Similarly $h_s^- \, g_t = g_t \, h_{se^t}^-$: negative horocycles are **expanded**.



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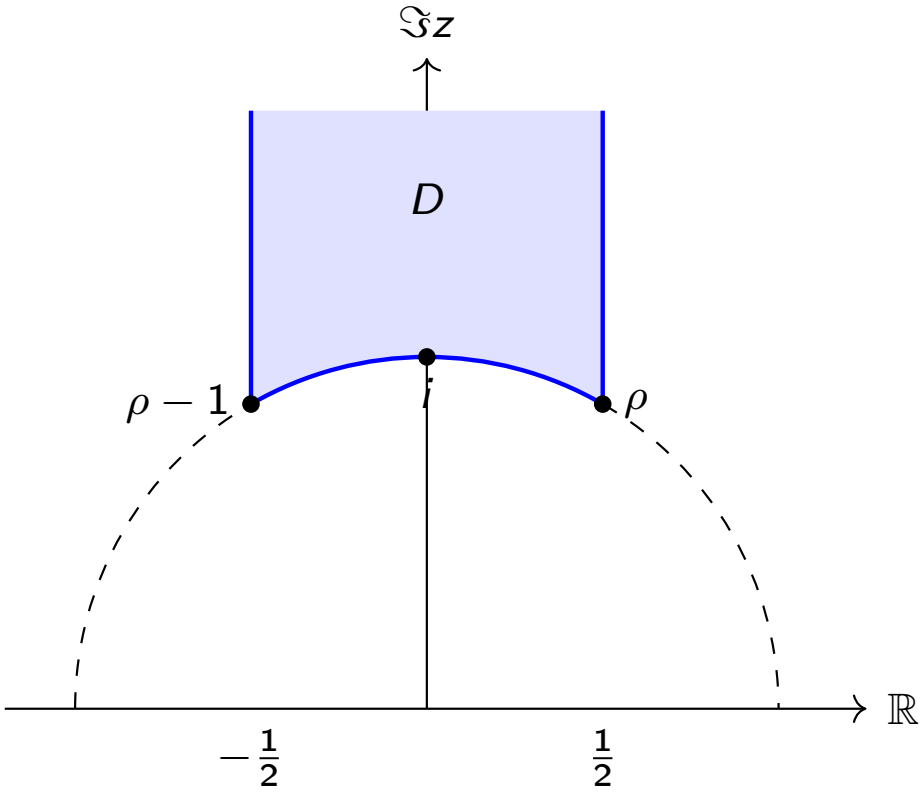
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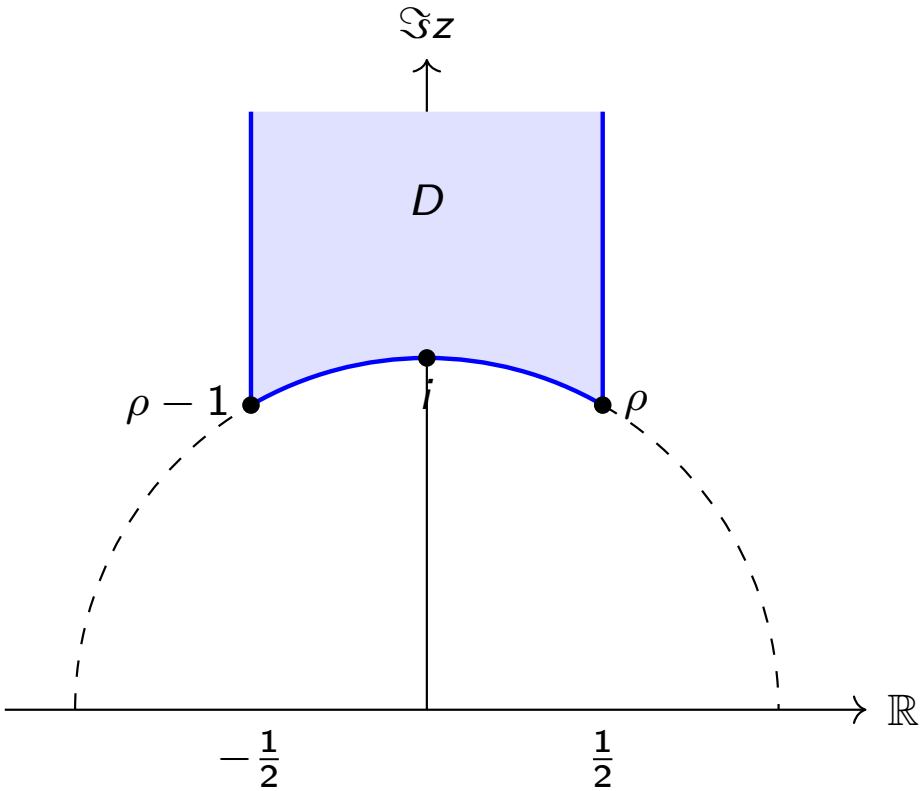
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A fundamental domain



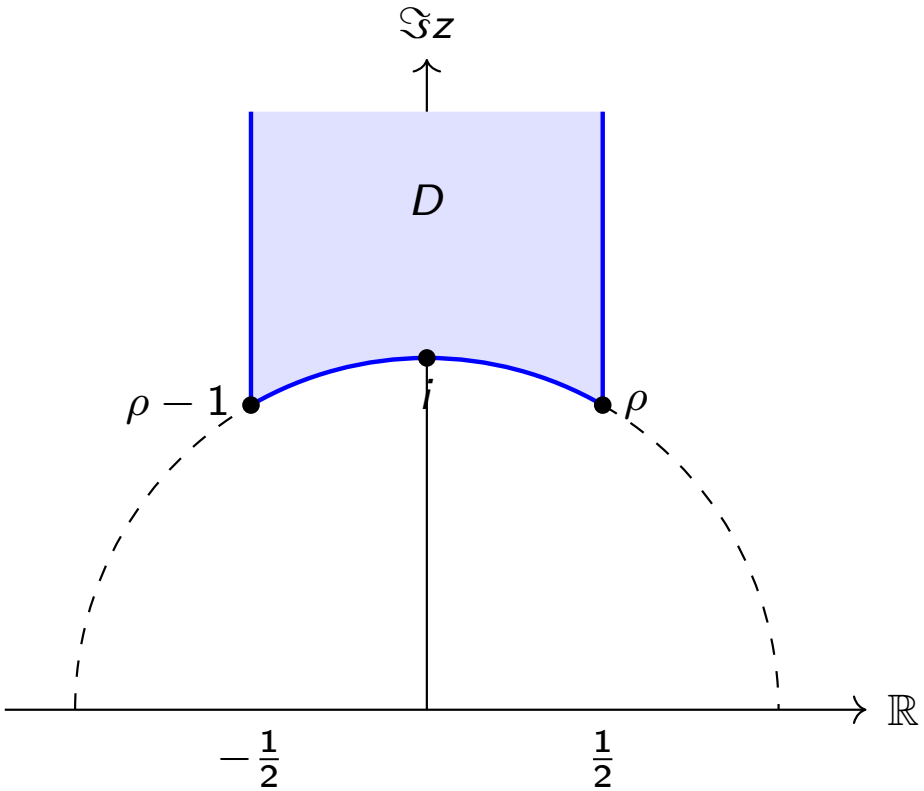
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- A neighborhood of the cusp is $\{\Im z > y_0\}/(z \mapsto z+1)$, of area $1/y_0$. What does a cusp look like?

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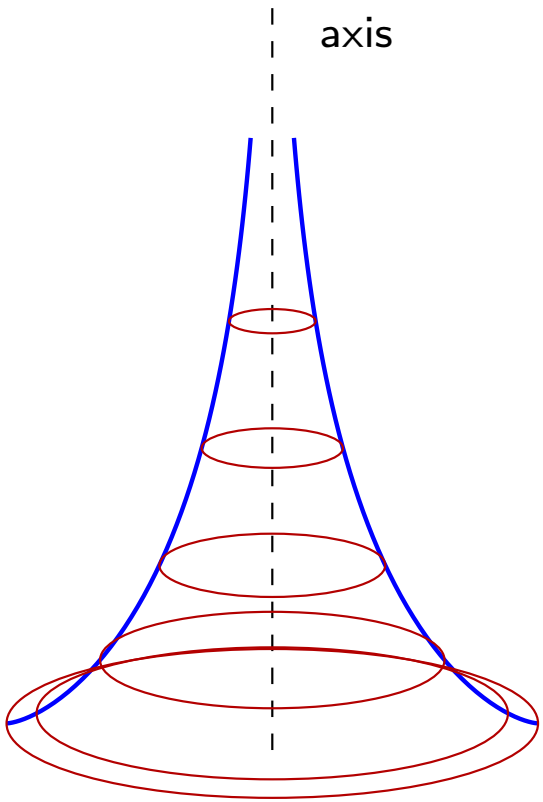
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The pseudosphere

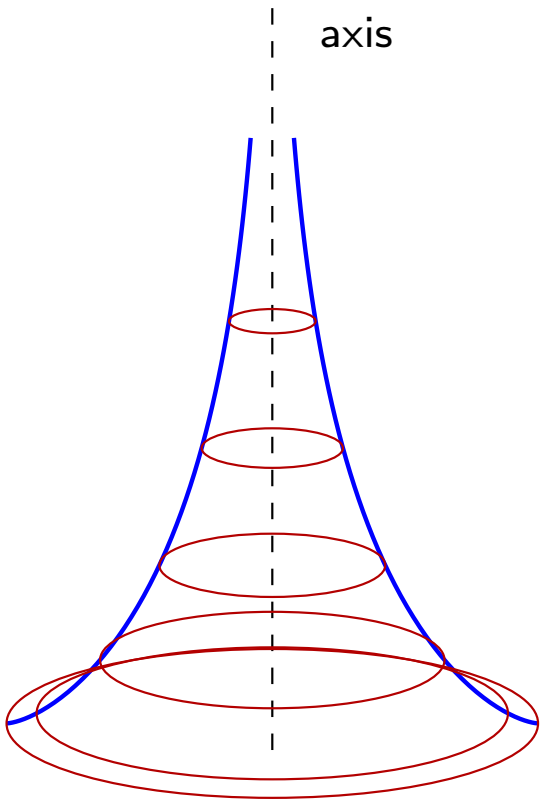


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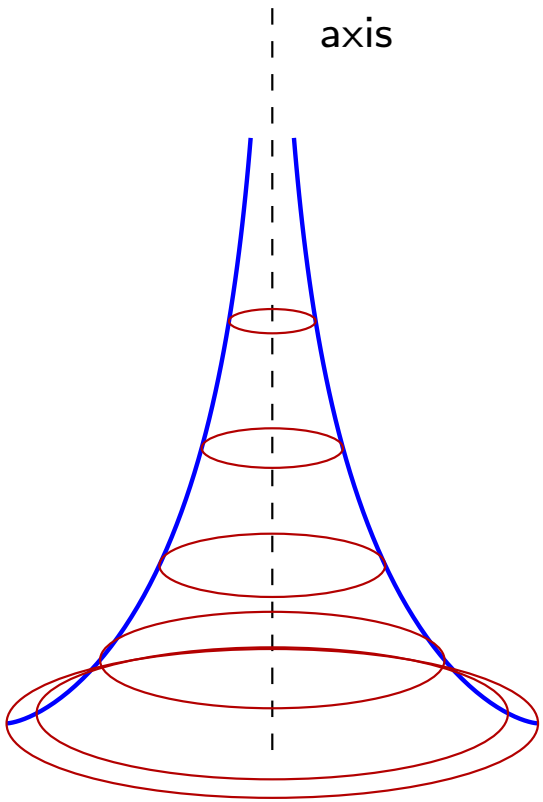
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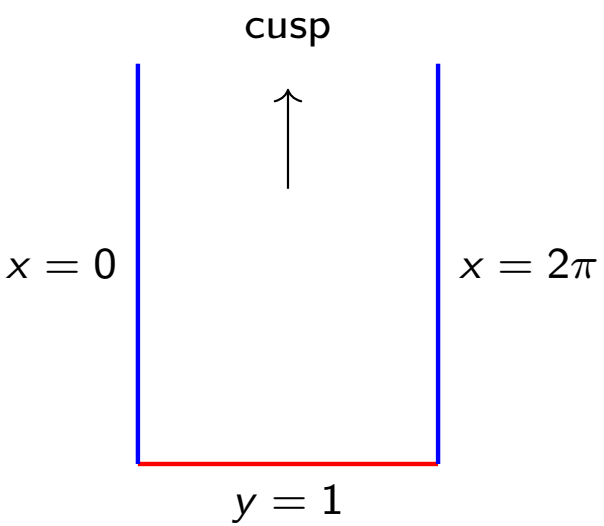
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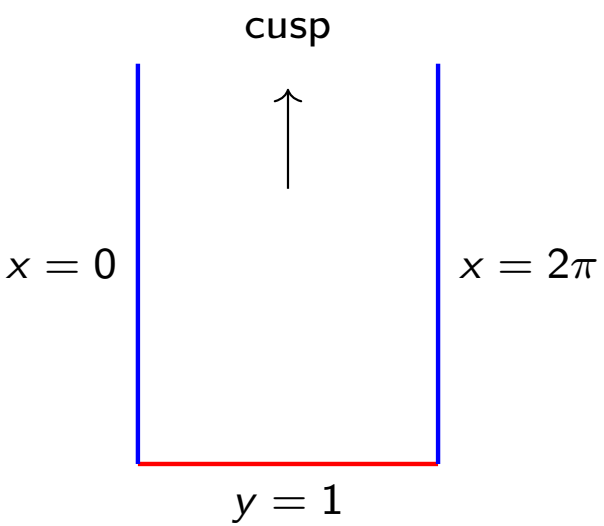
The pseudosphere is a piece of the hyperbolic plane



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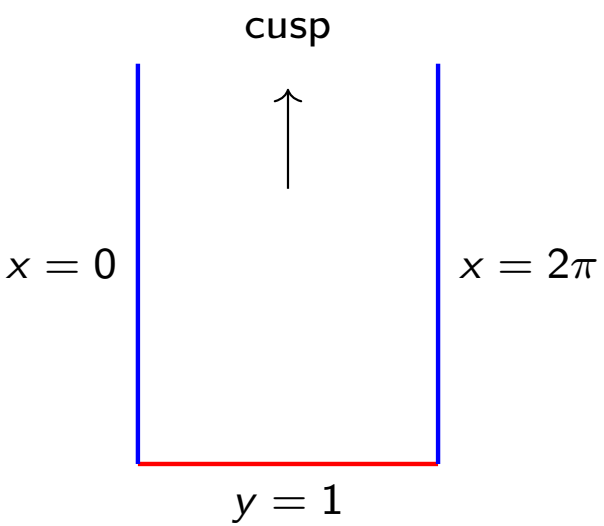


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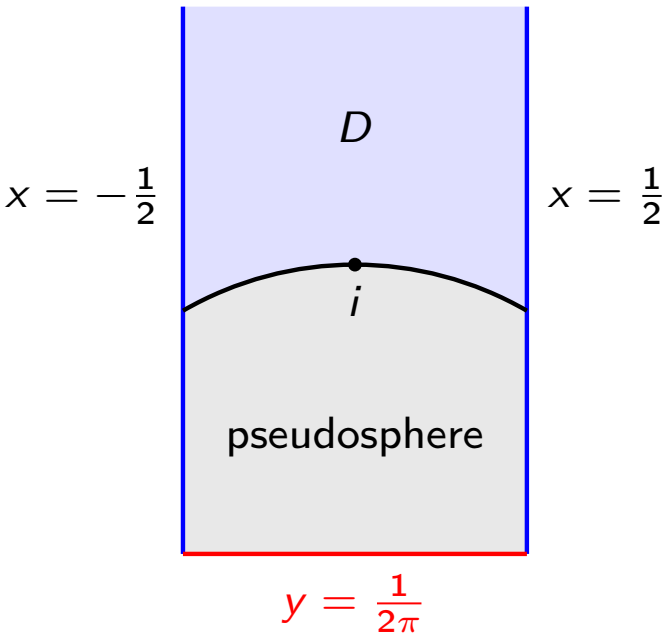


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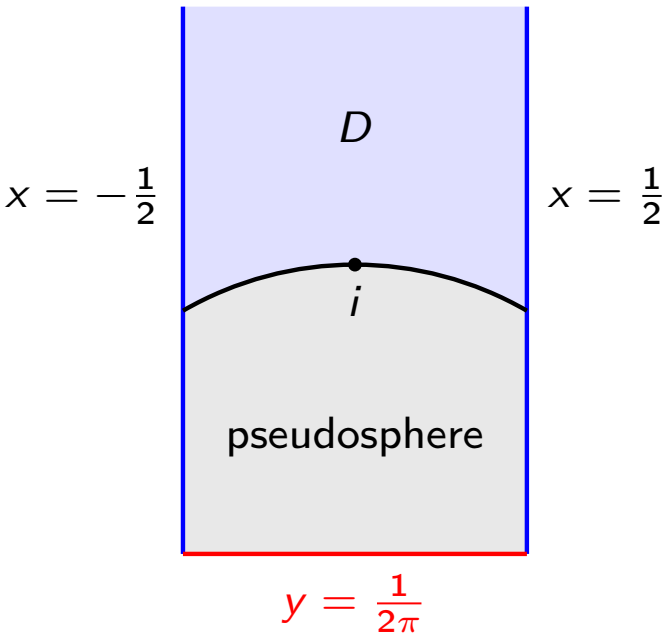
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- **Meridians** $\leftrightarrow$ vertical geodesics $x = \theta$. **Parallels** $\leftrightarrow$ horocycles $y = \cosh u$: they are closed, of length $\frac{2\pi}{\cosh u}$.

The pseudosphere as a strip



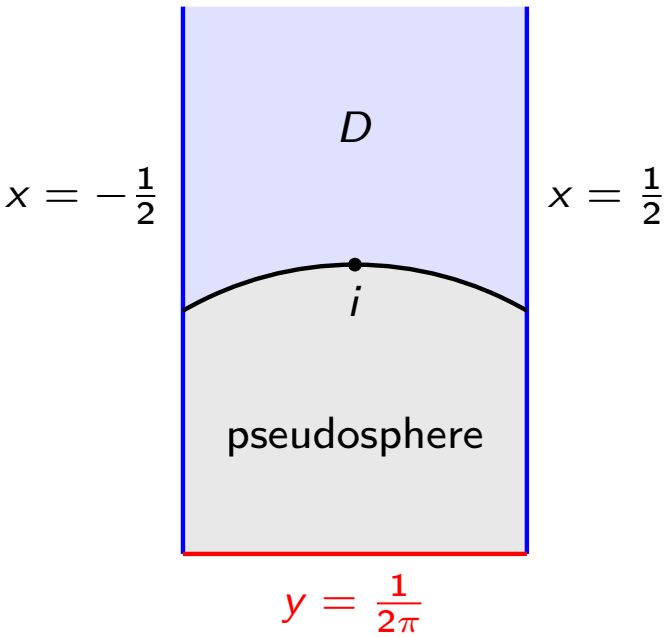
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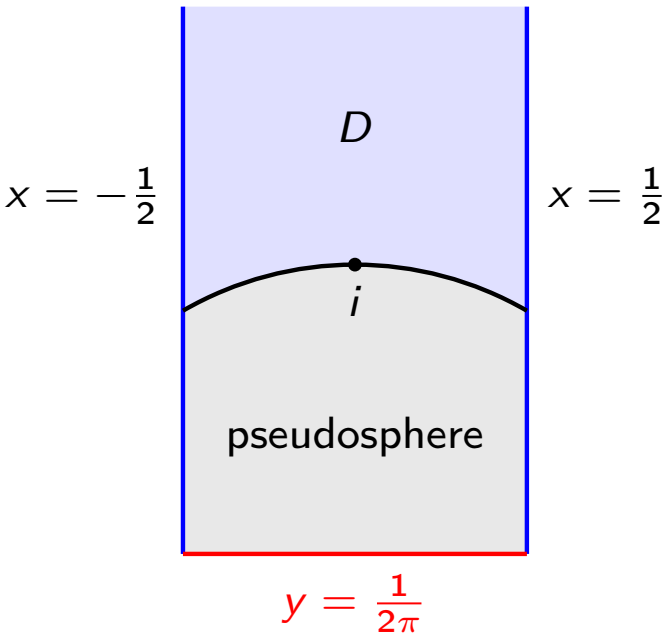
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- So the modular surface is the part of the pseudosphere above the arc $|z| = 1$, whose boundary is then folded by S.

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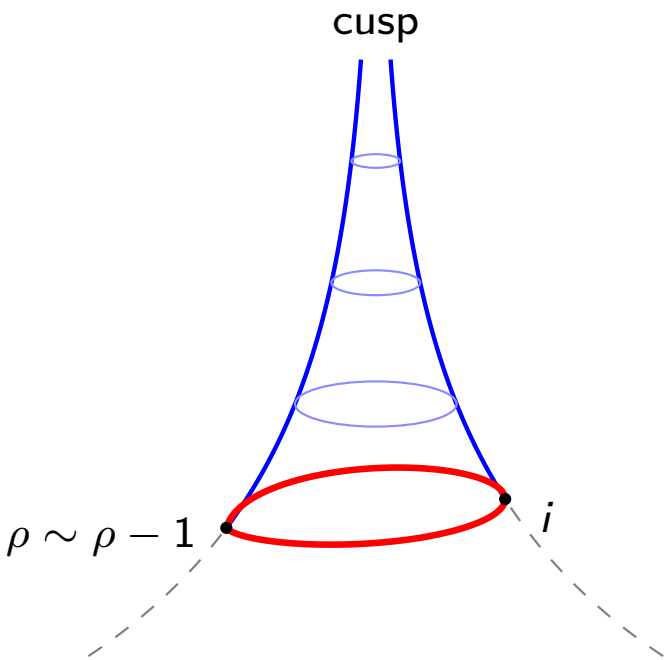
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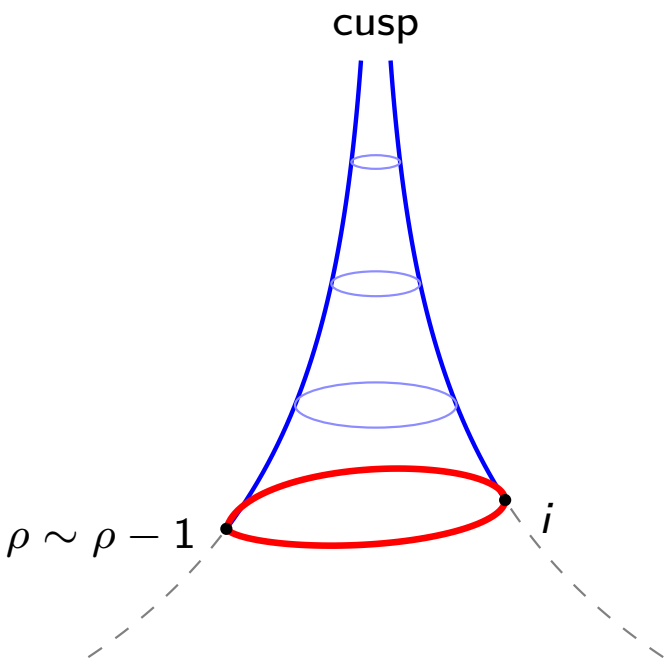
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- **Remark** (Hilbert, 1901): $\mathbb{H}$ cannot be realized in $\mathbb{R}^3$ as a complete surface of curvature -1: the pseudosphere has a singular rim, and models only the end.

A perspective view of the modular surface



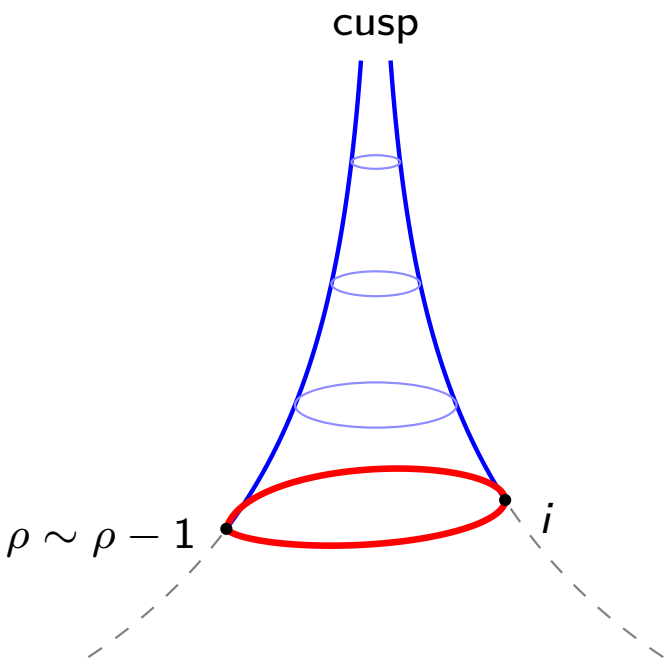
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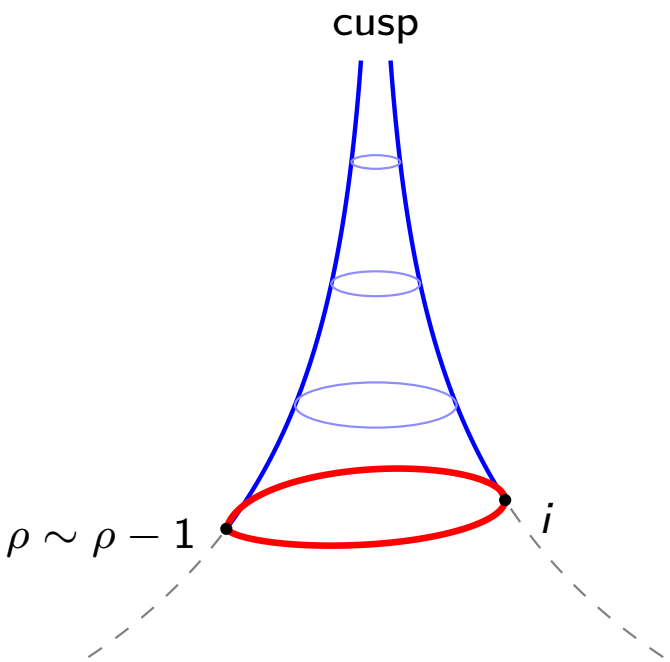
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- No identification is drawn. Not to scale: horizontal distances $\times 4$.

The flows on the modular surface

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- The geodesic flow preserves this measure: it is a measure preserving flow on the unit tangent bundle of the modular surface. The same is true for the horocycle flows.

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- For the last theorems we need a stronger assumption: $\Gamma\backslash\mathbb{H}$ is **compact**. This is not the case for the modular surface, because of the cusp; slightly weaker versions of these theorems hold for it.

Exercises (1/4): the hyperbolic plane

- **Exercise 1**. Show that $\Im(M \cdot z) = \dfrac{\Im z}{|cz+d|^2}$, and that $(MM') \cdot z = M \cdot (M' \cdot z)$.
- **Exercise 2**. Show that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}$, and that the stabilizer of i is $\mathrm{SO}(2,\mathbb{R})$.
- **Exercise 3**. Find the geodesic through i and $2+i$, and compute the hyperbolic distance $d(i, 2+i)$.
- **Exercise 4**. Compute the hyperbolic area of the ideal triangle with vertices 0, 1, ∞.
- **Exercise 5** ($\star$). Show that $\cosh d(z,w) = 1 + \dfrac{|z-w|^2}{2\,\Im z\,\Im w}$.

Exercises (2/4): flows and the modular group

- **Exercise 6**. Verify the fundamental property $h_s^+ g_t = g_t h_{se^{-t}}^+$, and find the corresponding relation for h_s^-.
- **Exercise 7**. Describe the orbits of g_t, h_s^+ and h_s^- through the vertical unit vector at $x_0 + iy_0$.
- **Exercise 8**. Show that $\mathrm{SL}(2,\mathbb{Z})$ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$, and that it acts properly discontinuously on $\mathbb{H}$.
- **Exercise 9** ($\star$). Show that every point of $\mathbb{H}$ is equivalent under $\mathrm{SL}(2,\mathbb{Z})$ to a point of $D = \{|x| \leq \frac{1}{2},\ |z| \geq 1\}$.
- **Exercise 10**. Show that the horocycle $\{y = y_0\}$ of the modular surface has length $1/y_0$ and that the region $\{y > y_0\}$ has area $1/y_0$ ($y_0 \geq 1$).

Exercises (3/4): the pseudosphere

- **Exercise 11**. Show that the surface of revolution of the tractrix with a leash of length a has curvature $-1/a^2$.
- **Exercise 12**. Compute the area of the pseudosphere, and the length of the parallel at height u. Show that the meridians have infinite length.
- **Exercise 13**. Show that the arc $|z| = 1$ of the fundamental domain corresponds, on the pseudosphere, to the curve $\cosh u = \sqrt{4\pi^2 - \theta^2}$, $|\theta| \leq \pi$.

Exercises (4/4): closed geodesics

- **Exercise 14** ($\star$). Let $M \in \mathrm{SL}(2,\mathbb{R})$ with $|\mathrm{tr}\, M| > 2$. Show that M preserves a geodesic of $\mathbb{H}$, on which it acts by a translation of length $2\,\mathrm{arccosh}\big(|\mathrm{tr}\, M|/2\big)$. Compute the length of the closed geodesic of the modular surface associated with $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and compare it with the golden ratio.
- **Exercise 15** ($\star$). Show that any geodesic which is not a vertical line is completely determined by its extremities x_+ and x_- (be careful that geodesic are oriented). If γ is a geodesic, and A an element of $\mathrm{SL}(2,\mathbb{Z})$, what are the extremities of $A\gamma$? Suppose that the geodesic g crosses the usual fundamental domaine for $\mathrm{SL}(2,\mathbb{Z}$; what can we say of the signs of x_+ and x_- ? Is it possible in this case to compute the continued fraction of the extremities of $A\gamma$ for A corresponding to $-1/z$ or $z-1$? Can you see a possible link to the Gauss map?