Examples in dynamical systems
Interval exchanges, Rauzy induction and Arnoux–Rauzy sequences

Pierre Arnoux

Lecture 9

From rotations to interval exchanges

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- The space of lattices $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ is replaced by the moduli space of translation surfaces; the Gauss map and the continued fraction are replaced by the induction of interval exchanges (Rauzy, Veech).
- We study here the dynamics of a single interval exchange: minimality criteria (Keane).

Interval exchange transformations: definition

- **Definition**: cut $[0,1[$ into k intervals $I_i = [x_{i-1}, x_i[$, with $0 = x_0 < x_1 < \cdots < x_k = 1$, and rearrange them by translations: $T(x) = x + \tau_i$ for $x \in I_i$, so that T is a **bijection** of $[0,1[$.

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- The I_i are the **continuity intervals**; $D = \{x_1, \ldots, x_{k-1}\}$ is the set of **discontinuities**.
- $k = 2$: $T(x) = x + \lambda_2 \bmod 1$ is a **rotation**. Interval exchanges generalize rotations.

Coordinates (λ, π)

- **Lengths**: $\lambda_i = |I_i| = x_i - x_{i-1}$, so $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Delta_k = \{\lambda \in \mathbb{R}^k_{>0} : \sum \lambda_i = 1\}$.

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- **Permutation**: the images $T(I_i)$ are intervals; call $J_1 < \cdots < J_k$ the images, ordered from left to right. Then $T(I_i) = J_{\pi(i)}$, where π is a permutation of $\{1, \ldots, k\}$: the rank of the image.

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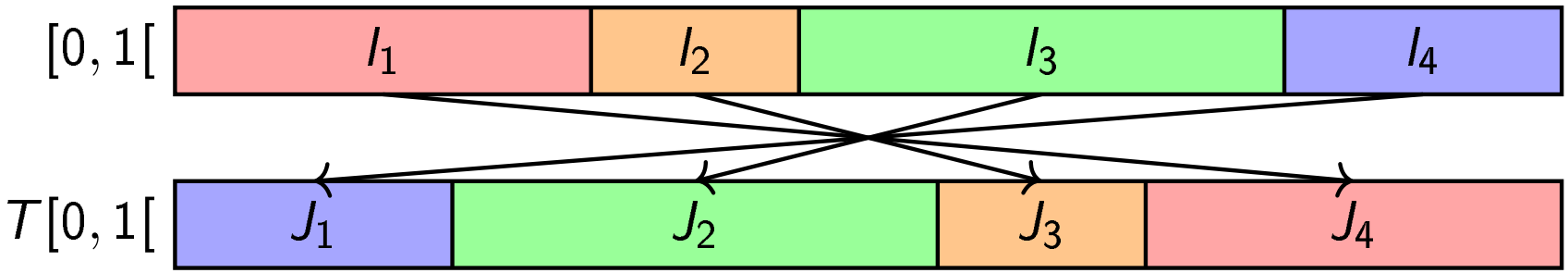
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- So $T = (\lambda, \pi)$ is completely determined by $\lambda \in \Delta_k$ and $\pi \in \mathfrak{S}_k$.

Example: four intervals, the reversing permutation $\pi(i) = 5 - i$



$\tau_1 = \lambda_2 + \lambda_3 + \lambda_4$, $\tau_2 = \lambda_3 + \lambda_4 - \lambda_1$, $\tau_3 = \lambda_4 - \lambda_1 - \lambda_2$,
$\tau_4 = -(\lambda_1 + \lambda_2 + \lambda_3)$.

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- **Reducible permutations**: if $\pi(\{1, \ldots, n\}) = \{1, \ldots, n\}$ for some $n < k$, then $[0, x_n[$ is invariant: T is never minimal. We assume from now on that π is **irreducible**: $\pi(\{1, \ldots, n\}) \neq \{1, \ldots, n\}$ for all $n < k$.

Keane's lemma: induction of an interval exchange

- Let T be an interval exchange on k intervals, and $I = [a, b[\subset [0, 1[$. For $x \in I$, let $n(x) = \inf\{n > 0 : \ T^n(x) \in I\}$ be the first return time.

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- **Lemma (Keane)**: the first return map $T_I(x) = T^{n(x)}(x)$ is defined everywhere on I, the return time is bounded, and T_I is again an **interval exchange**, with **at most** $k + 2$ **intervals**; the return time is constant on each of them.

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- Same idea as the induction of rotations: the induced system belongs to the same family.
- **Remark**: a good choice of I keeps exactly k intervals: Rauzy induction, the analogue of the Euclidean algorithm for interval exchanges.

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- These points cut I into at most $k + 2$ semi-open intervals J_i. On each of them $n(x) = n_i$ is constant and the iterates T^j, $j \leq n_i$, are translations.
- So the return time is defined and bounded ($\max n_i$), and T_I is a bijection of I which is a translation on each J_i: an interval exchange with at most $k + 2$ intervals. ■

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- **Theorem (Keane)**: T is **minimal** (every orbit is dense) if and only if it satisfies the Keane condition.
- In words: there is no invariant finite union of intervals with endpoints in the orbits of the discontinuities.

The Keane condition: idea of the proof

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- ($\Leftarrow$) *Step 1*: aperiodicity and finite measure imply that D^∞ is **dense** in $[0,1]$: otherwise, by finite measure, some T^p would map a gap of D^∞ into itself by a translation, hence by the identity, and its points would be periodic.

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- *Step 2*: take $I = [a, b[$ with $a, b \in D^\infty$. By Keane's lemma, the union F of the *towers* $T^j(J_i)$, $0 \leq j < n_i$, is a finite union of semi-open intervals with endpoints in D^∞, and $T(F) \subset F$. By conservation of measure, $T(F) = F$: so $F = [0,1[$.

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- *Step 3*: so the backward orbit of every point meets I. The same holds for T^{-1}, which satisfies the condition too: every forward orbit meets every I with endpoints in D^∞, which are dense: T is minimal. ■

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- **Corollary**: for an irreducible π, almost every $\lambda \in \Delta_k$ (for Lebesgue measure) gives a minimal interval exchange: the exceptional set is a countable union of hyperplanes.
- We prove the theorem by showing that the i.d.o.c. holds.

Proof of the irrationality criterion (1/2): a relation

- **Veech's matrix**: the translations are $\tau = M^\pi \lambda$, where $M^\pi_{c,m} = [\pi(c) > \pi(m)] - [c > m]$ is an antisymmetric integer matrix ($+1$ if $c < m$ and $\pi(c) > \pi(m)$; -1 if $c > m$ and $\pi(c) < \pi(m)$; 0 otherwise).

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$$\sum_{\ell=0}^{n-1} (M^\pi \lambda)_{i_\ell} = T^n(x_i) - x_i = x_j - x_i = \sum_m \big([m \leq j] - [m \leq i]\big)\lambda_m.$$

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- This is an **integer linear relation** $\sum_m c_m \lambda_m = 0$, with $c_m = \sum_{\ell=0}^{n-1} M^\pi_{i_\ell,m} - [m \leq j] + [m \leq i]$.

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- Let $a_m = \#\{\ell \in [1, n] : \ i_\ell > m\}$ and $b_m = \#\{\ell \in [0, n-1] : \ \pi(i_\ell) > m\}$. By the formula for $M^\pi_{c,m}$, $c_m = b_{\pi(m)} - a_m$:

$$\sum_{m=1}^{k} \big(b_{\pi(m)} - a_m\big)\lambda_m = 0.$$

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- Let $N = \max\{m : \ a_m > 0\}$: then $1 \leq N < k$, and $\{m : \ a_m > 0\} = \{1, \ldots, N\}$. Since b is nonincreasing, $\{m' : \ b_{m'} > 0\} = \{1, \ldots, N'\}$, and $a_m = b_{\pi(m)}$ gives $\pi(\{1, \ldots, N\}) = \{1, \ldots, N'\}$, so $N' = N$.

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- So $\pi(\{1, \ldots, N\}) = \{1, \ldots, N\}$ with $N < k$: π is reducible. Contradiction: the i.d.o.c. holds, and T is minimal. ■

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- **Ergodic measures**: a minimal interval exchange on k intervals has at most $k-1$ ergodic measures (they are linearly independent, and $\mu \mapsto \lambda_\mu$ is linear and injective into Δ_k); in fact at most $[k/2]$ (the form M^π is antisymmetric), and it is uniquely ergodic for $k = 3$.

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- Next: the induction (Rauzy–Veech) as a dynamical system on the space of interval exchanges, a higher dimensional version of the Gauss map.

Exercises on interval exchanges (1/2)

- **Exercise 5**. Show that $\tau_i = \sum_{\pi(j)<\pi(i)} \lambda_j - \sum_{j<i} \lambda_j$.
- **Exercise 6**. Show that an interval exchange on two intervals is conjugate, by a bijection, to a rotation of the circle; find its angle in terms of λ_1, λ_2.
- **Exercise 7**. Show that T^{-1} is an interval exchange; give its discontinuities, its lengths and its permutation. Show that T^n has at most $n(k-1)+1$ intervals of continuity.
- **Exercise 8**. Show that an interval exchange preserves the Lebesgue measure, and that, if $\pi(\{1,\dots,n\}) = \{1,\dots,n\}$ for some $n < k$, it is not minimal.

Exercises on interval exchanges (2/2)

- **Exercise 9**. Show that a periodic point of an interval exchange lies in an interval of points of the same period, whose left endpoint is in the orbit of a discontinuity.
- **Exercise 10** ($\star$). Show that if the orbits of the discontinuities are infinite and distinct, then T satisfies the Keane condition.
- **Exercise 11**. For the four intervals and the reversing permutation of the example, find a vector λ with $T^n(x_i) = x_j$ for some $n \geq 1$, and check the relation $\sum(a_m - b_{\pi(m)})\lambda_m = 0$.
- **Exercise 12**. Show that every irrational rotation is uniquely ergodic.

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- If $\alpha_1 = \alpha_2$, then f is ν-a.e. constant, so $\mu_1 = \mu_2$. Hence $\alpha_1 \neq \alpha_2$: take $E_i = \{f = \alpha_i\}$. ■

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- **Lemma**: if $\mu_1, \ldots, \mu_n$ are distinct ergodic invariant probabilities, there are sets $E_1, \ldots, E_n$ with $\mu_i(E_j) = \delta_{ij}$, so functions $\phi_i = 1_{E_i}$ with $\int \phi_i \, d\mu_j = \delta_{ij}$.

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- **Proof**: for $i \neq j$, take E_{ij} with $\mu_i(E_{ij}) = 1$, $\mu_j(E_{ij}) = 0$ (previous lemma), and $E_i = \bigcap_{j \neq i} E_{ij}$.

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- **Corollary 1**: $\mu_1, \dots, \mu_n$ are **linearly independent** in the space of measures: if $\sum c_j \mu_j = 0$, integrating ϕ_i gives $c_i = 0$.
- **Corollary 2**: if there is only one ergodic measure, there is only one invariant probability (every invariant probability is a mixture of ergodic ones: ergodic decomposition).

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- Hence $h(x)=\mu([0,x[)$ is continuous (no atom) and strictly increasing (full support): a **homeomorphism** of $[0,1[$, and $h_*\mu=$ Lebesgue measure.

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- So T_μ is an interval exchange. Since h is increasing, the order of the intervals and of their images is unchanged: $T_\mu = (\lambda_\mu, \pi)$ with the **same permutation** π, and $\lambda_\mu = \big(\mu(I_1), \ldots, \mu(I_k)\big)$.

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- Then $g(T^n 0) = T^n(g(0)) = T^n 0$: g is the identity on the orbit of 0, which is dense by minimality. So $g = \mathrm{id}$, $h_\mu = h_\nu$, and $\mu = \nu$.

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- So the invariant measures of a minimal T embed linearly in $\mathbb{R}^k$, as vectors with positive coordinates summing to the mass.

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- **Theorem**: a minimal interval exchange on k intervals has at most $k-1$ ergodic measures.
- **Proof**: suppose there are k of them, $\mu_1, \ldots, \mu_k$. The vectors λ_{μ_i} (normalized) form a basis of $\mathbb{R}^k$ in the hyperplane $\sum x_i = 1$: their convex hull contains a nonempty open set of this hyperplane, hence a rational vector $\lambda' = \lambda_\mu$ for some invariant μ (a convex combination of the μ_i). Contradiction with the Lemma. ■

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- Weyl equidistribution follows: for every x and every continuous f, $\frac{1}{n}\sum_{j<n} f(R^j x) \to \int f$.
- The next question: what happens for $k \geq 3$?

The antisymmetric matrix M^π and the bound $[k/2]$

- Veech's matrix $M^\pi_{ij} = 1$ if $i < j,\ \pi(i) > \pi(j)$; -1 if $i > j,\ \pi(i) < \pi(j)$; 0 otherwise. It is antisymmetric and $\tau = M^\pi \lambda$. We assume $\pi(n+1) \neq \pi(n) + 1$.

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- Let V be the vector space spanned by the vectors λ_μ (μ invariant), $N = \ker M^\pi$. Then $V \cap N = \{0\}$: if $M^\pi \mu = 0$ the two measures give conjugate IETs with the same translation vectors, so $\mu = 0$.

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- The Birkhoff sums of $\tau(x) = \tau_i$ on I_i are bounded ($S_n \tau(x) = T^n x - x$), so $\int \tau \, d\mu' = \langle \lambda', \tau \rangle = 0$. Applied to $T_{\mu''}$, whose translation vector is $M^\pi \lambda''$: $\langle \lambda', M^\pi \lambda'' \rangle = 0$ on V. V **is isotropic** for the antisymmetric form.

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- But **almost every** (λ, π) is uniquely ergodic (Masur, Veech 1982): this is the main result of the next two sections.

Rauzy induction

Induction again: Euclid's algorithm

- For $k = 2$, $T = (\lambda, (2\,1))$ is the rotation of Lecture 3. Induce on $[0, 1 - \delta[$ with $\delta = \min(\lambda_1, \lambda_2)$: we get again an exchange of two intervals, of lengths $(\lambda_1 - \lambda_2, \lambda_2)$ or $(\lambda_1, \lambda_2 - \lambda_1)$.

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- Iterating gives a "multidimensional continued fraction", a dynamical system on the space of interval exchanges.

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- Label the intervals by letters. The **top row** lists the intervals in the order of the domain $I_1 < \cdots < I_k$, the **bottom row** lists the *same letters* in the order of their images $J_1 < \cdots < J_k$. The permutation is $\pi = \pi_b \circ \pi_t^{-1}$.

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- Let α_t be the **last letter of the top row** and α_b the **last letter of the bottom row**. As π is irreducible, $\alpha_t \neq \alpha_b$.

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Rauzy induction: the rule

- Compare λ_{α_t} and λ_{α_b} (equality is excluded by Keane's condition: it would be a connection between orbits of discontinuities). Induce T on $[0, 1-\delta[$, $\delta = \min(\lambda_{\alpha_t}, \lambda_{\alpha_b})$.

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- Example: from $\begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$, if $\lambda_C > \lambda_A$ we get $\begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$; if $\lambda_A > \lambda_C$ we get $\begin{pmatrix} A & C & B \\ C & B & A \end{pmatrix}$.

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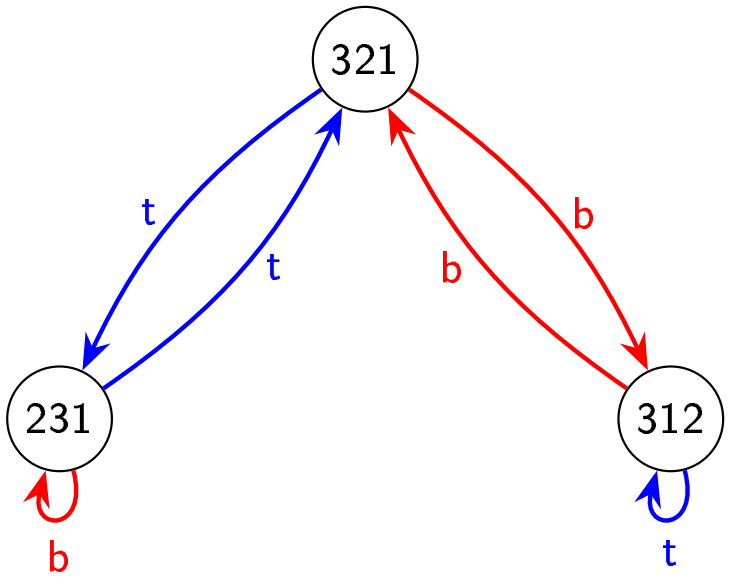
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- Bottom wins: same argument for T^{-1} (exchange the roles of the two rows). ■

Rauzy induction

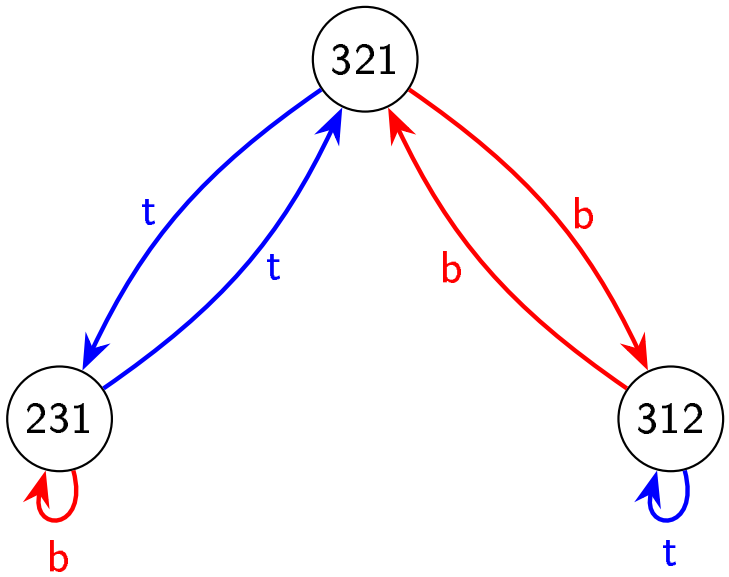
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- **Rauzy class** of π: the permutations reached from π by top and bottom moves. The **Rauzy diagram** has these vertices and two arrows (t and b) out of each.

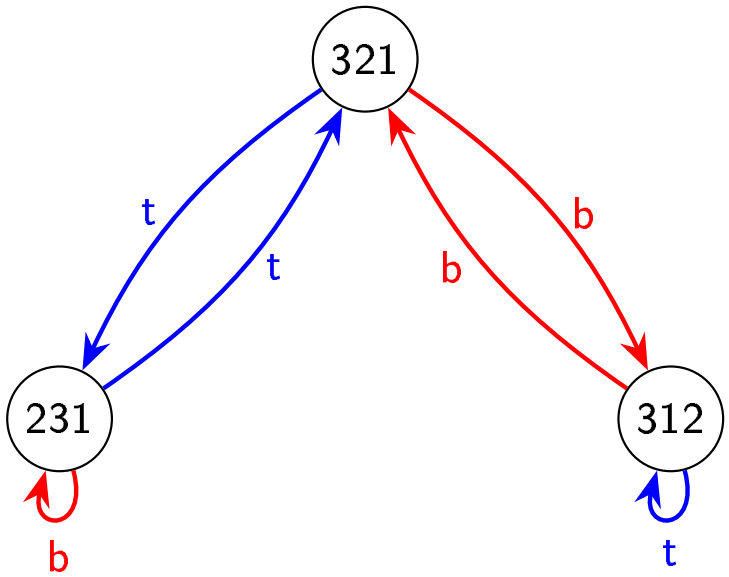
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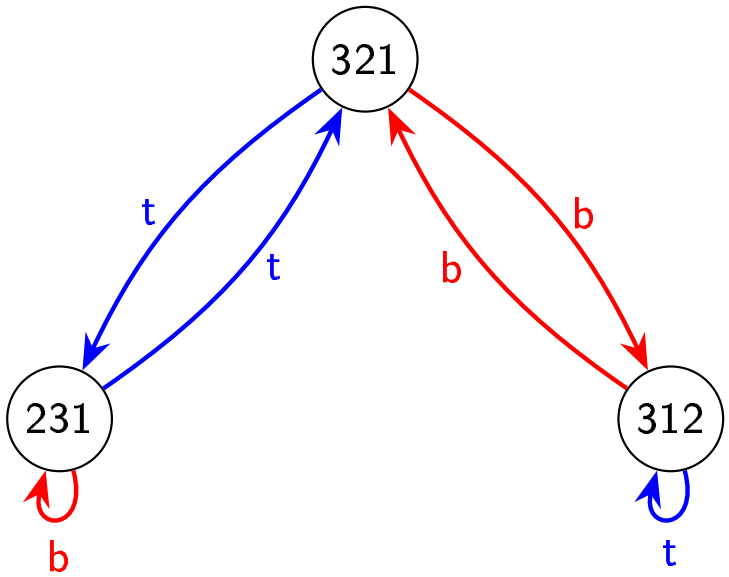
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- A **path** in the diagram records which letter wins at each step: these are the "digits".

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- Use positions in the top row, and $b = \pi^{-1}(k)$ (the interval whose image is last). One step gives $\lambda = B\lambda'$, with B a nonnegative integer matrix of determinant 1 depending only on the step in the diagram:

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- For $k = 2$: $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: the Euclid matrices of Lecture 3.

The Rauzy–Veech map

- Normalize: $R(\lambda, \pi) = \left(\frac{B^{-1}\lambda}{|B^{-1}\lambda|}, \ \pi' \right)$ on $\Delta_k \times \mathcal{R}$, where $\mathcal{R}$ is the Rauzy class: the analogue of the Farey map ($k = 2$).

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- The "digits" are paths in a finite diagram, instead of integers.

Cylinders and cones

- For a path γ of length n starting at π_0, the set of (λ, π_0) whose expansion begins with γ is

$$[\gamma] = \{\lambda \in \Delta_k : \ B_\gamma^{-1}\lambda > 0\} = B_\gamma(\mathbb{R}_+^k) \cap \Delta_k :$$

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- A nonempty **open** set: it has positive Lebesgue measure.
- The images $B^{(n)}(\mathbb{R}_+^k)$ are **nested cones**: $B^{(n+1)}\mathbb{R}_+^k = B^{(n)}B_{n+1}\mathbb{R}_+^k \subset B^{(n)}\mathbb{R}_+^k$ because B_{n+1} is nonnegative.

Acceleration and the Gauss measure

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- The analogy: for the Farey map the invariant measure is dx/x, infinite, conservative and ergodic; for the Gauss map $\frac{dx}{(1+x)\ln 2}$, finite.
- Conservative and ergodic give: *almost every point visits infinitely often every set of positive measure*. This is the only property of the measure we shall use.

Invariant measures follow the Rauzy path of T

- Let $T = (\lambda, \pi)$ be minimal and satisfy the i.d.o.c.: its Rauzy expansion is infinite, with matrices $B_1, B_2, \ldots$ and induced maps $T^{(n)}$ with lengths $\lambda^{(n)} > 0$, $\lambda = B^{(n)}\lambda^{(n)}$.

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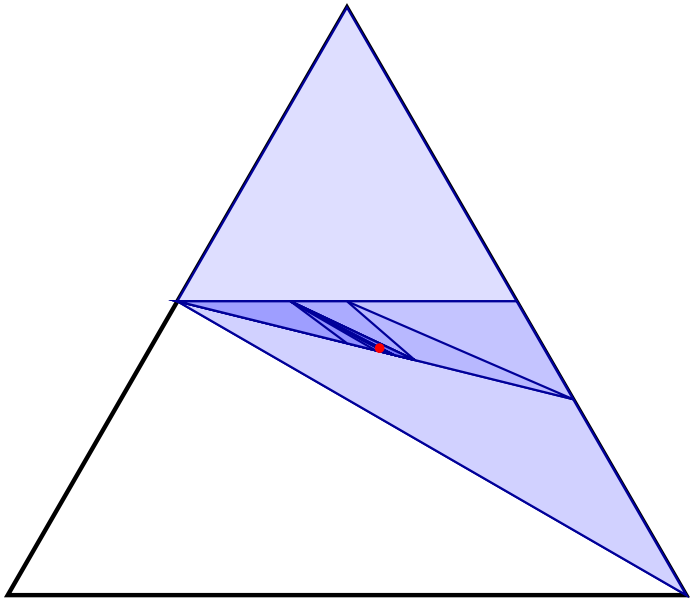
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- **Proof**: $T_\mu = hTh^{-1}$ with h increasing. Which of λ_{α_t}, λ_{α_b} is larger is decided by an *order* comparison between two points of $[0, 1[$ (x_{k-1} and the left end of J_k): h preserves it. The induced maps are conjugate by h, and we iterate.

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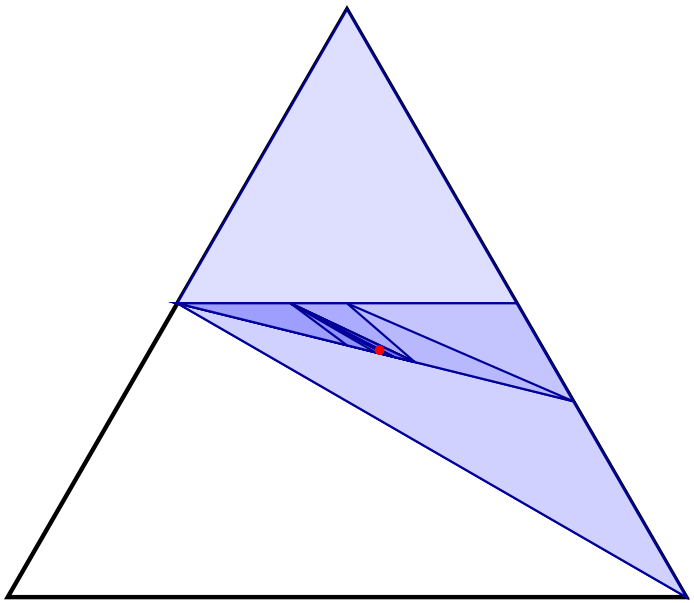
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- Hence $\lambda_\mu = B^{(n)}\lambda_\mu^{(n)}$ with $\lambda_\mu^{(n)} > 0$: $\lambda_\mu \in C_n := B^{(n)}\mathbb{R}_+^k$ for all n.

The invariant measures lie in a nested intersection of cones



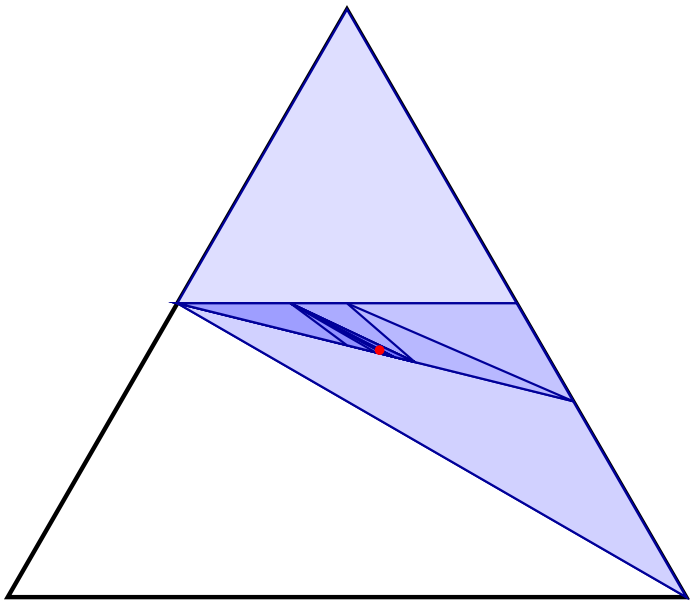
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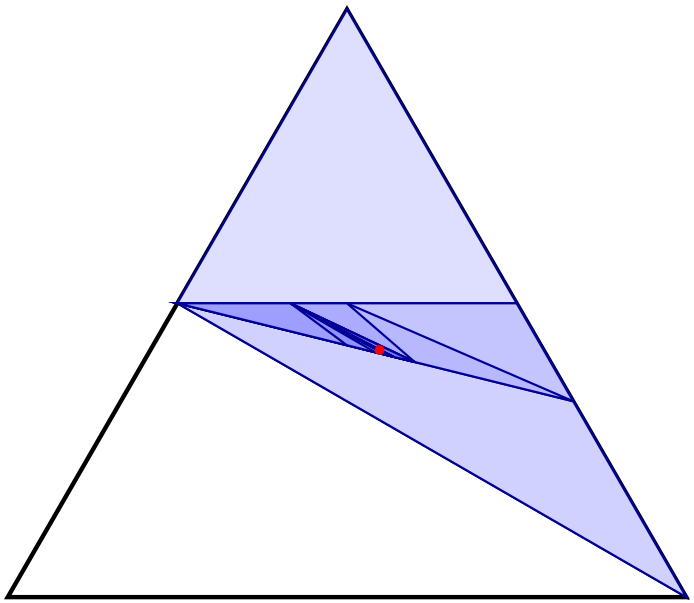
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- So we must show that the cones shrink to a ray, for almost every λ: **Perron–Frobenius**.

Hilbert metric and positive matrices

- On the open orthant $\mathbb{R}^k_{>0}$, Hilbert's projective metric: $d_H(x,y) = \log \max_{i,j} \frac{x_i y_j}{x_j y_i}$ (a distance between rays).

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- Permuting the coordinates does not change $\Delta(A)$.

Infinitely many positive blocks: the cones shrink to a ray

- Suppose the product $B_1 B_2 B_3 \cdots$ contains, infinitely often, disjoint blocks equal to a fixed positive matrix P:
  $B^{(n)} = Q_0 P Q_1 P Q_2 P \cdots$ with Q_j nonnegative invertible.

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- Nested compact cones whose Hilbert diameter tends to 0: $\bigcap C_n$ is a **single ray**.

The Veech–Masur theorem

A positive loop in the Rauzy diagram

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$$B_\gamma = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 2 \end{pmatrix}, \qquad \det B_\gamma = 1.$$

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- For $k = 4$ the shortest positive loop at the reversing permutation has length 11 (found by a computer search). The general existence is admitted.

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Almost every expansion contains the loop infinitely often

- The cylinder $[\gamma] \subset \Delta_k \times \{\pi_0\}$ of the loop is a nonempty open set: it has positive measure, for Lebesgue and for the equivalent Gauss measure.

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- Each visit at time n_j means that the matrices $B_{n_j+1} \cdots B_{n_j+|\gamma|}$ are those of the loop: choosing visits separated by more than $|\gamma|$ steps, we get **infinitely many disjoint blocks equal to** B_γ.

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- Each visit at time n_j means that the matrices $B_{n_j+1} \cdots B_{n_j+|\gamma|}$ are those of the loop: choosing visits separated by more than $|\gamma|$ steps, we get **infinitely many disjoint blocks equal to** B_γ.
- (Ergodicity, not only conservativity, is needed: otherwise some paths could never enter $[\gamma]$.)

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- By Perron–Frobenius, $\bigcap_n C_n$ is a ray. The invariant measures lie in this ray, and $\mu \mapsto \lambda_\mu$ is injective: there is at most one invariant probability, and Lebesgue measure is one. ■

The Veech–Masur theorem

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- The loop γ replaces the positive blocks $M_i M_j^n M_k$ of Arnoux–Rauzy in the next section.

Complexity and special factors

- u a minimal sequence, $p(n)$ the number of factors of length n. For a factor U, let $\partial^+ U \geq 1$ be the number of letters α with $U\alpha$ a factor, and $\partial^- U$ the number of letters β with βU a factor. Then

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- **Sturmian sequences**, $p(n) = n+1$ (Lecture 2, the rotation coding): exactly one right special factor of each length, with 2 extensions; and one left special.
- The graph Γ_n of the factors of length n (an edge for each factor of length $n+1$): two loops glued along a segment; each loop is a *segment name*, and these names evolve by τ_1, τ_2, the substitutions of Euclid's algorithm.

Complexity $2n+1$: the condition $(*)$

- If $p(n) = 2n+1$ then $p(n+1) - p(n) = 2 = \sum(\partial^+ U - 1)$: at each length, **either** one right special factor with 3 extensions, **or** two right special factors with 2 extensions. Same for the left.

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- **Evolution**: from n to $n+1$, either the graph is *slit* (the segment between G_n and D_n is shortened; names unchanged), or it *bursts* ($G_n = D_n$): then, up to the choice of a letter, $(U, V, W) \mapsto (U,\ VU,\ WU)$.

The three substitutions and the directing sequence

- Define substitutions on $\{1,2,3\}$:

$$\begin{aligned}
\sigma_1 &: \quad 1 \mapsto 1, \quad 2 \mapsto 21, \quad 3 \mapsto 31 \\
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- Why each value infinitely often: otherwise the system contains a periodic sequence, or, if only two values occur, it contains a system of complexity $n + C$ (a Sturmian-like system): contradiction with minimality.
- This is the abstract "continued fraction" of the sequence u: the analogue of the proposition of Lecture 2 for Sturmian sequences.

The Tribonacci sequence

- The Tribonacci substitution $\tau : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ has the fixed point $u = 1213121121312\ldots$ The periodic directing sequence $(1,2,3)^\infty$ gives it: $\sigma_1\sigma_2\sigma_3$ maps $1 \mapsto 1213121 = \tau^3(1)$, $2 \mapsto 213121$ (the mirror image of $\tau^3(2) = 121312$) and $3 \mapsto 3121$ (the mirror image of $\tau^3(3) = 1213$). So $\sigma_1\sigma_2\sigma_3$ is the *mirror image* of τ^3, and the language is closed under reversal.

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- **Periodic directing sequence** (analogue of Lagrange's theorem): the system is generated by a primitive substitution on 3 letters, with dominant eigenvalue a **cubic** algebraic number Θ, and the interval lengths lie in $\mathbb{Q}(\Theta)$.

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- Some of them are weakly mixing (Cassaigne–Ferenczi–Messaoudi 2006): a very different behaviour from Sturmian sequences.

A simultaneous continued fraction

- The three matrices of the abelianization (columns: letters), the inverses of the *subtractive* steps $(\alpha, \beta, \gamma) \mapsto (\alpha - \beta - \gamma, \beta, \gamma)$ and its permutations:

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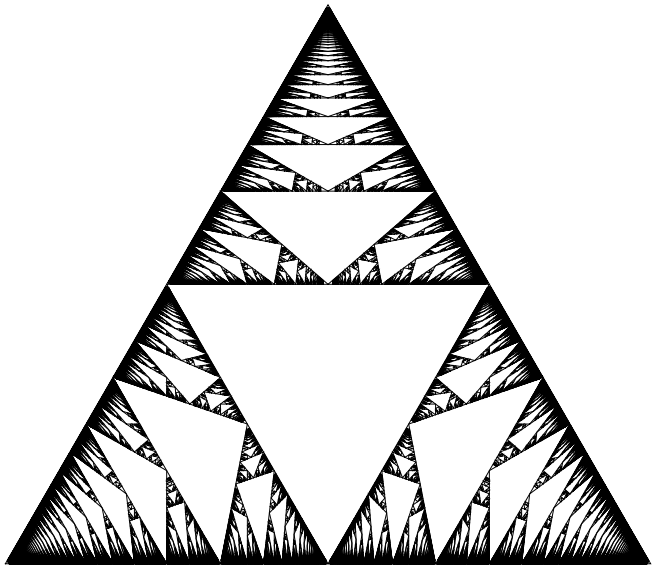
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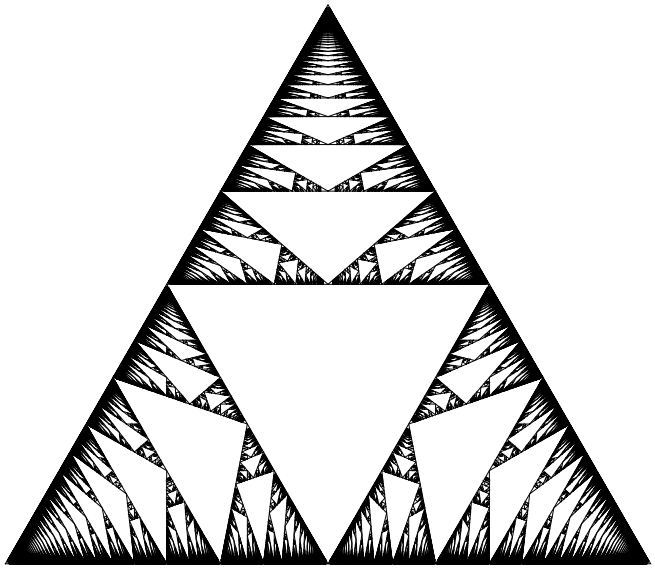
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- The expansion of (α, β, γ) is possible only if, at each step, **one weight exceeds the sum of the other two**.

The Rauzy gasket



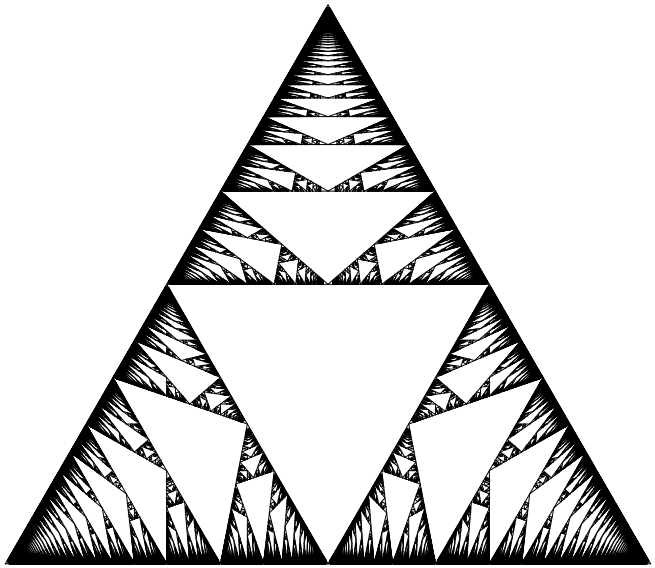
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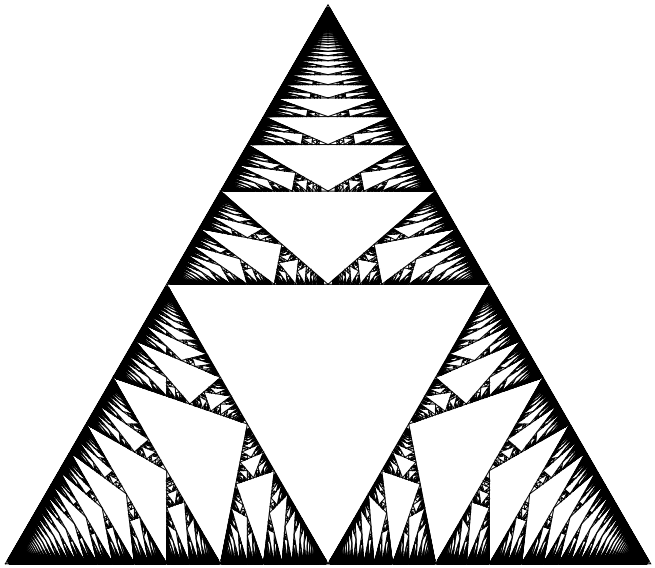
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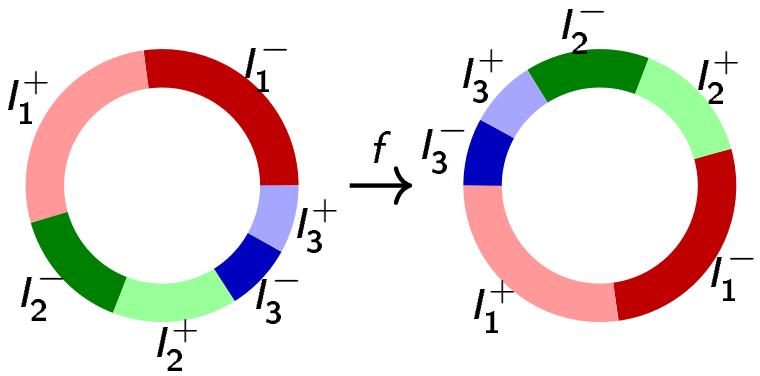
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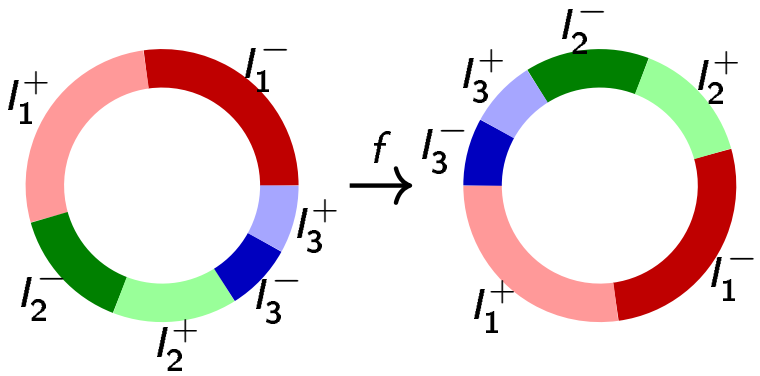
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- As Arnoux–Rauzy noted in 1991: one can induce indefinitely only on a set of measure zero, of the Sierpiński carpet type.

The six interval exchange



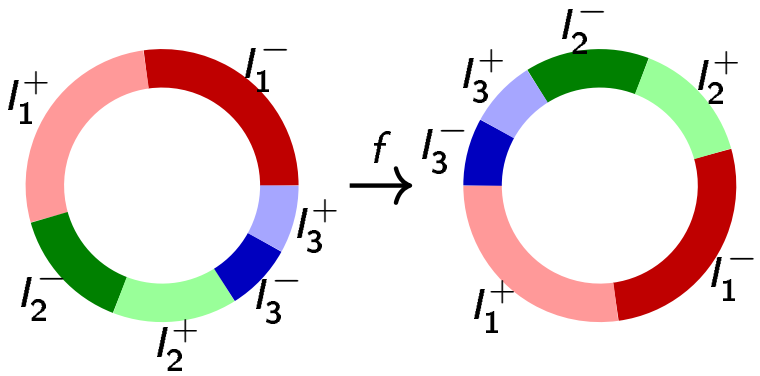
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- Left: the six arcs; right: their images (same colours). f is a 6-interval exchange on the circle, with lengths $\frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \frac{\gamma}{2}$: only 3 parameters.

Induction on an arc: the same family

- **Lemma** (Arnoux–Rauzy 1991): let f' be the first return map of f to $f(I_1)$, whose ends are identified. Then f' is an exchange of the same type, with weights $(\alpha', \beta', \gamma') = (\alpha - \beta - \gamma,\ \beta,\ \gamma)$.

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- The same holds for σ_2 and σ_3, inducing on the image of I_2 or I_3, when $\beta > \alpha + \gamma$ or $\gamma > \alpha + \beta$.

The representation theorem

- Iterate: given (i_k), choose (α, β, γ) in the nested intersection of cones (a ray). At each step one weight exceeds the sum of the other two, so we can induce on the corresponding arc (by σ_{i_k}). The arcs $J^{(k)} = \bigcup J_i^{(k)}$ shrink to three points, the common ends of I_1, I_2, I_3.

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- Remark: the natural coding of this exchange by its *six* arcs is a different sequence: it has complexity $6n$.
- The weights of the exchange lie in the **Rauzy gasket**: the construction works exactly for the parameters of the gasket.

A thin family of exchanges, and its dynamics

- The six lengths $\left(\frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\beta}{2}, \frac{\gamma}{2}, \frac{\gamma}{2}\right)$ span a $\mathbb{Q}$-vector space of dimension $3 < 6$: not totally irrational, and, as the gasket has measure 0, **the Veech–Masur theorem says nothing** about this family.

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- The sequences themselves are uniquely ergodic; the six-interval exchanges are not always: there are non-uniquely ergodic examples (Dynnikov–Skripchenko). Under a Diophantine condition on the directing sequence, they are uniquely ergodic and isomorphic to the sequence (Arnoux–Cassaigne–Ferenczi–Hubert 2019).

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- For the periodic sequence $(1,2,3)^\infty$ (Tribonacci) the 6-exchange is **self-induced** and the system is a rotation of the torus $\mathbb{T}^2$ (Rauzy 1982): the subject of the last slides. This holds for a large class (Arnoux–Ito 2001) and almost all AR systems (Berthé–Steiner–Thuswaldner 2019), but not all.

Tribonacci: three models of one system

- Let $\theta \simeq 1.8393$, $\theta^3 = \theta^2 + \theta + 1$. The Tribonacci system has three realizations:
  - the **symbolic system**: the fixed point of $1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$;
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  - the **translation** $R_a : x \mapsto x + a$ of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, with $a = (\theta^{-1}, \theta^{-2})$.

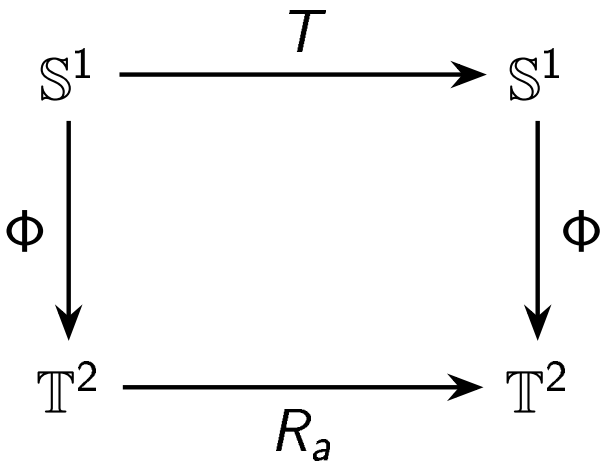
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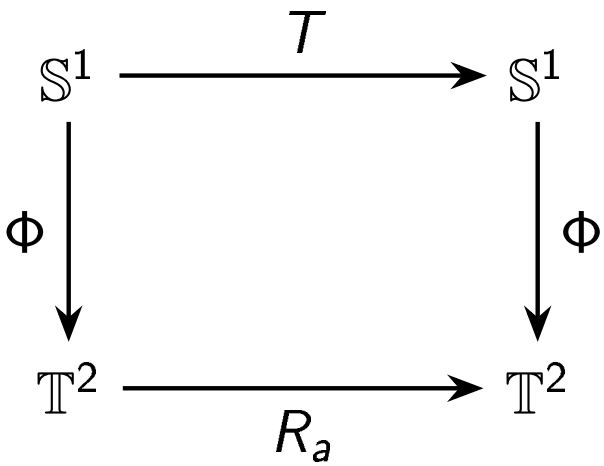
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  Arnoux–Cassaigne–Ferenczi–Hubert (2019): T is measure-theoretically isomorphic to R_a.
- Arnoux (1988): the isomorphism is realized by a **continuous** map, a **plane-filling curve**.

The semi-conjugacy



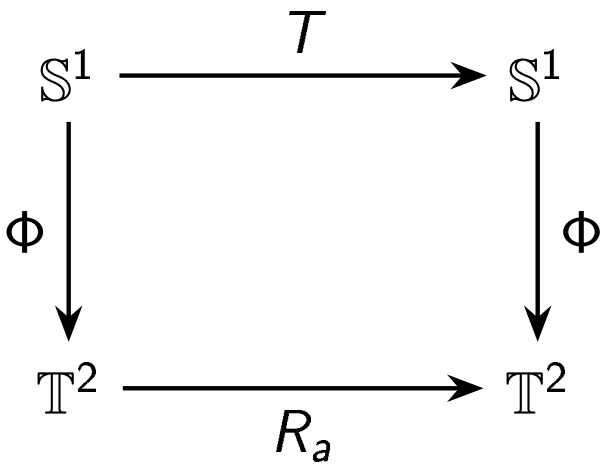
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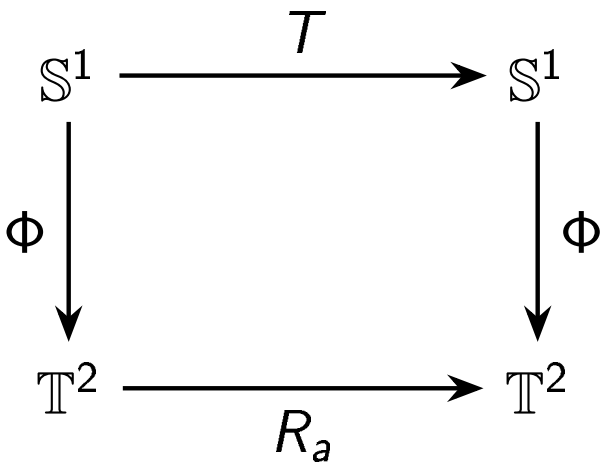
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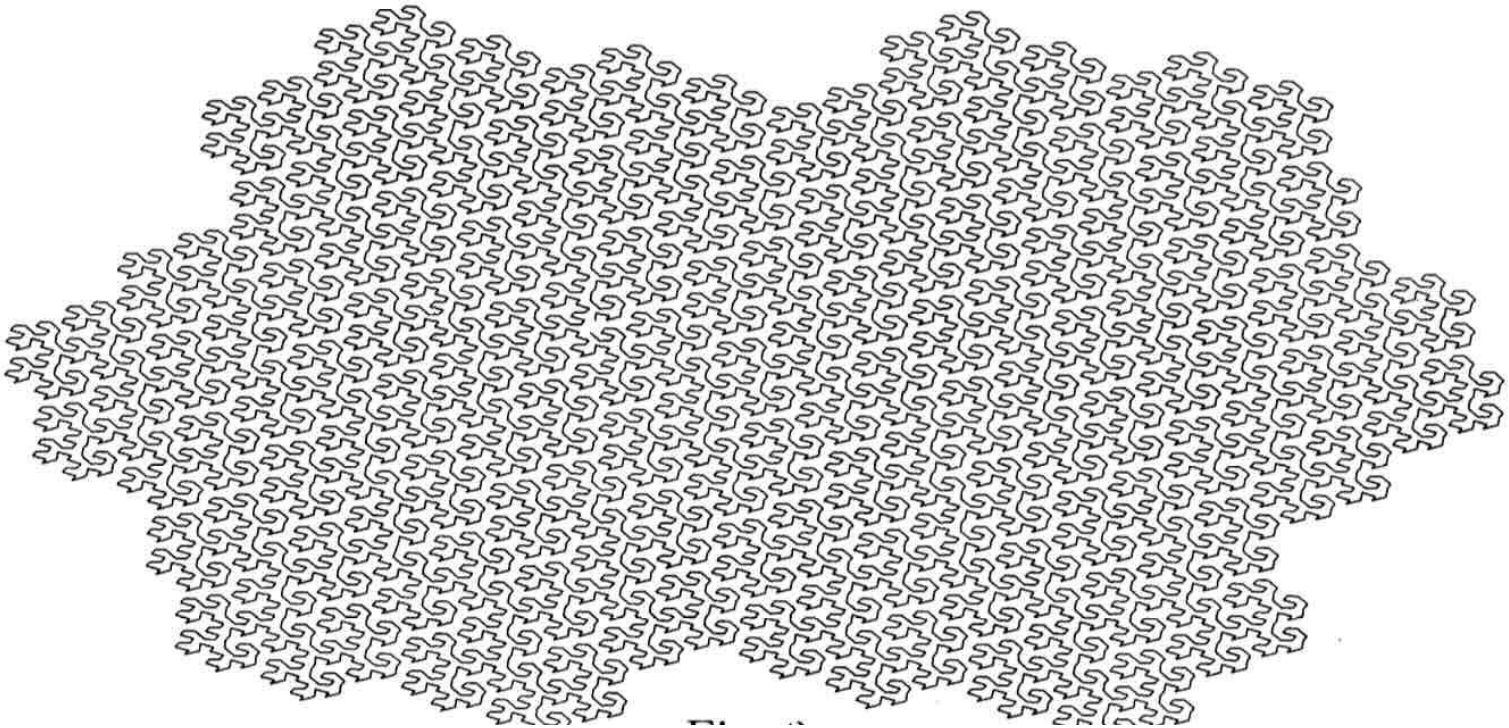
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- The images of the six arcs are tiles with fractal boundary, which tile the torus: Rauzy's fractal domains.

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- The images of the six arcs are tiles with fractal boundary, which tile the torus: Rauzy's fractal domains.
- A sequence of curves, built by substitution, converges to Φ (picture).

The plane-filling curve



Approximation of the plane-filling curve Φ (lifted to the plane): figure from P. Arnoux, *Un exemple de semi-conjugaison entre un échange d'intervalles et une translation sur le tore*, Bull. SMF 116 (1988).