CIMPA Course

Basic Part

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Dynamical Systems — Lecture Notes

Dynamical Systems: Basic Definitions

A **dynamical system** is a pair (X, f), where:

- X is a metric space,
- $f : X \to X$ is a continuous map.

Metric spaces

A set $X \neq \emptyset$ is called a **metric space** if there exists a function

$$d : X \times X \to \mathbb{R}_{\geq 0} \quad \text{(a metric)}$$

satisfying the following:

(i) $d(x, y) = d(y, x)$ *(Symmetry)*

(ii) $d(x, y) = 0 \iff x = y$ *(only the diagonal Δ of $X \times X$ maps to 0)*

(iii) $d(x, z) \leq d(x, y) + d(y, z), \quad \forall\, x, y, z \in X$ *(Triangle inequality)*

We say (X, d) is a metric space.

Continuity

Is $f : X \to X$ continuous? We use **pointwise continuity**.

$x_0 \in X$ is a point of continuity if

$$\forall\, \varepsilon > 0,\ \exists\, \delta(x_0, \varepsilon) > 0 \ \text{ such that } \ d(f(x), f(x_0)) < \varepsilon \ \text{ whenever } \ d(x, x_0) < \delta.$$

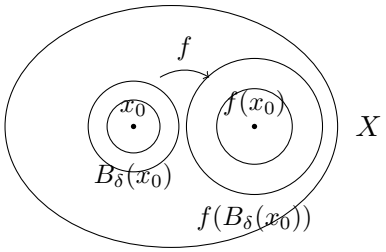
Define the open ball

$$B_\varepsilon(x) = \{\, y \in X \mid d(x, y) < \varepsilon \,\}.$$

Continuity (equivalent formulation).

$$\forall\, x \in B_\delta(x_0) \implies f(x) \in B_\varepsilon(f(x_0))$$

$$\iff \quad f\big(B_\delta(x_0)\big) \subseteq B_\varepsilon\big(f(x_0)\big).$$



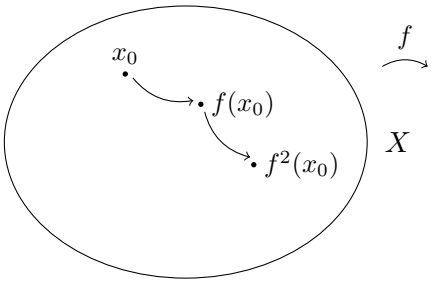
Dynamical Notions

Forward orbit

Let $x_0 \in X$ (the **initial condition**). The **forward orbit** of x_0 under f is defined as

$$O^+(x_0) = \{\, f^n(x_0) \mid n \in \mathbb{N} \cup \{0\} \,\}, \qquad f^0(x_0) = I(x_0) = x_0.$$

The forward orbit may be **finite** or **infinite**.



Fixed points

$x_0 \in X$ is called a **fixed point** if

$$f(x_0) = x_0 \quad \Longleftrightarrow \quad f^n(x_0) = x_0 \;\; \forall\, n.$$

If x_0 is a fixed point, then $O^+(x_0) = \{x_0\}$.

Periodic points

$x_0 \in X$ is a **periodic point** if there exists $k \in \mathbb{N}$ such that

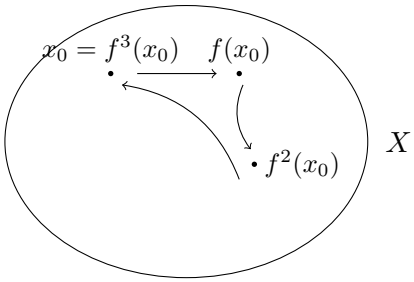
$$f^k(x_0) = x_0.$$

If x_0 is a periodic point, the **period** is

$$k = \min\{\, n \mid f^n(x_0) = x_0 \,\}.$$

$\Longrightarrow$ x_0 has finite orbit:

$$O^+(x_0) = \{\, x_0,\; f(x_0),\; \ldots,\; f^{k-1}(x_0) \,\}.$$



If x_0 is not periodic, x_0 is called an **aperiodic point**, and the orbit is infinite.

Homeomorphisms and Backward Orbits

If $f : X \to X$ is a **homeomorphism**: $f : X \to X$ is continuous + bijective + $f^{-1} : X \to X$ is continuous.

For $x_0 \in X$, the set

$$O(x_0) = \{\, f^n(x_0) \mid n \in \mathbb{Z} \,\}$$

is called the **full orbit**.

Preimages

For $x_0 \in X$,

$$f^{-1}(x_0) = \{\, x \in X \mid f(x) = x_0 \,\}$$

is a set. For a bijective map, $f^{-1}(x_0)$ is a **singleton**.

Backward orbit

$$O^-(x_0) = \{\, f^n(x_0) \mid n \in \mathbb{Z}_{\leq 0} \,\}$$

($\mathbb{Z}_{\leq 0}$: all non-positive integers.)

$O^-(x_0)$ is the **backward orbit** of x_0.

$$O^+(x_0) \cup O^-(x_0) = O(x_0).$$

In general, we will take our bijective map f enough to discuss the questions related to forward orbit.

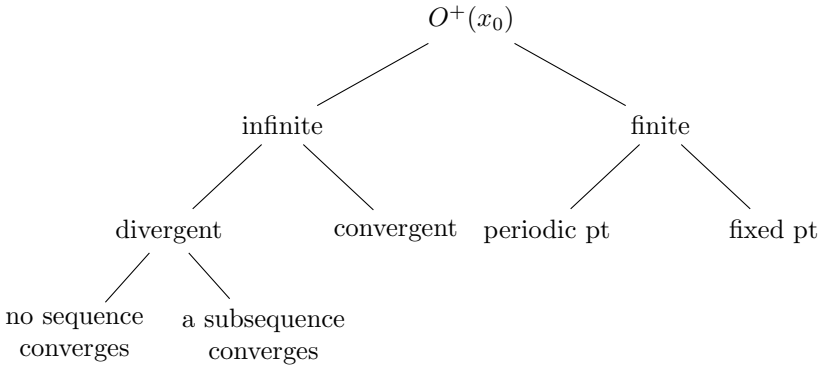
One-Dimensional Linear Dynamical Systems

1. Questions Related to Dynamical Systems

For (X, f) a dynamical system, we will investigate:

- Either there exists a fixed point or periodic points.
- For aperiodic points, which have infinite orbit — what is the behaviour of orbits (infinite) of an aperiodic point x_0?
- Either its orbit is convergent or divergent?
- If convergent, then what will be the limit of the orbit, where does it stay?
- If divergent, then where does a big part of the orbit stay in space? (convergent subsequence)
- Either orbits are dense?

For $x_0 \in X$ an initial condition, the forward orbit $O^+(x_0)$ can be classified as follows:



dense $\longleftarrow$ infinite

2. One-Dimensional Linear Map

Let $X = \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = ax + b, \qquad a, b \in \mathbb{R}, \quad a \neq 0.$$

Then $(\mathbb{R}, f)$ is a dynamical system.

$\mathbb{R}$ is a metric space with metric (Exercise):

$$d(x, y) = |x - y|.$$

$$B_\epsilon(x_0) = \{x \in \mathbb{R} : |x - x_0| < \epsilon\} = (x_0 - \epsilon,\ x_0 + \epsilon) \quad \text{open intervals.}$$

Claim: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ is continuous.

Let $\epsilon > 0$.

$$|f(x) - f(x_0)| = |a(x - x_0)| = |a|\,|x - x_0|, \qquad |a| \neq 0.$$

We want

$$|a|\,|x - x_0| < \epsilon \quad \Longrightarrow \quad |x - x_0| < \epsilon/|a| = \delta.$$

So, given $\epsilon > 0$, there exists $\delta = \epsilon/|a|$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

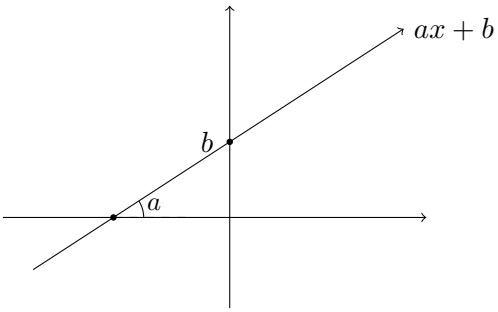
Therefore f is continuous, and so $(\mathbb{R}, f)$ forms a dynamical system.

3. Geometrical Representation of f

$$f(x) = ax + b$$

is an equation of a line with slope a and y-intercept b (slope-intercept form).

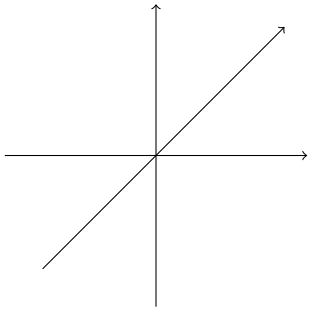
$$\text{Slope of line} = \tan\theta = a.$$



We will take $b = 0$; then

$$f_a(x) = ax, \qquad a \in \mathbb{R},$$

a line passing through the origin with slope a.



For each $a \in \mathbb{R}$, we have a dynamical system $(\mathbb{R}, f_a)$.

Therefore, $(\mathbb{R}, f_a)$ forms a **family of dynamical systems** parametrized by $a \in \mathbb{R}$.

4. Fixed Point

Let $f_a : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_a(x) = ax.$$

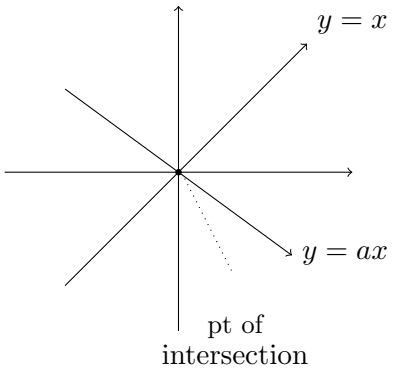
Then 0 is the only fixed point for all $a \in \mathbb{R}$:

$$f_a(0) = a(0) = 0, \qquad \forall a \in \mathbb{R}.$$

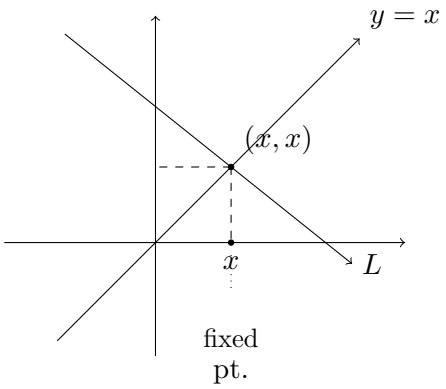
Geometrically, in $\mathbb{R}^2$, either two lines are parallel or they intersect at one point. $f_a(x) = ax$ is a line passing through the origin, which intersects the diagonal line $(y = x)$ at the origin.

$\Rightarrow (0, 0)$ is the point of intersection of the line $y = ax$ and $y = x$.

Therefore, 0 is the only fixed point.



Intuitively, the point of intersection of any line L in the plane with the diagonal line is the fixed point.



5. Iteration of Linear Systems

Let $(\mathbb{R}, f_a)$ be the one-dimensional linear system. Let $x_0 \in \mathbb{R}$ be the initial condition, then

$$f_a(x_0) = ax_0.$$

$$f_a^2(x_0) = f_a(f_a(x_0)) = f_a(ax_0) = a(ax_0) = a^2x_0.$$
$$f_a^3(x_0) = f_a(f_a^2(x_0)) = f_a(a^2x_0) = a^3x_0.$$

Therefore, in general,

$$f_a^n(x_0) = a^nx_0, \qquad \forall n \in \mathbb{N}.$$

Therefore, the orbit (trajectories) are a **geometric sequence**.

Regarding the convergence of $\{f_a^n(x_0) : n \in \mathbb{N}\}$, there are two possible cases:

(a) If $|a| < 1$

then $a \in (-1, 1)$, and

$$|f_a^n(x_0)| = |a^nx_0| = |a|^n|x_0| \xrightarrow{n\to\infty} 0 = f(0).$$

$\Rightarrow$ the orbit converges to the fixed point 0, for all $a \in (-1, 1)$.

(b) If $|a| > 1$

then

$$|f_a^n(x_0)| = |a|^n|x_0| \xrightarrow{n\to\infty} +\infty.$$

$\Rightarrow$ the orbit converges to $+\infty$ (divergent).

Case III: If $|a| = 1 \ \Rightarrow \ a = 1, \ a = -1$

If $a = 1$**:** $f_a(x) = x$, $\forall x \in \mathbb{R}$. Then every point is a fixed point.

If $a = -1$**:** $f_a(x) = -x$.

$$f_a^2(x) = f_a(f_a(x)) = f_a(-x) = x.$$

$\Rightarrow x$ is periodic with period 2.

Summary:

- For $a = 1$: every point $x \in \mathbb{R}$ is a fixed point.
- For $a = -1$: every point $x \in \mathbb{R}$ is periodic with period 2.

6. Geometric Way to Visualize the Orbit in One-Dimensional Linear Systems

The orbit of a one-dimensional linear dynamical system can be visualized in a simple geometric way (*cobweb diagram*).

Step I. From point $x_0 \in \mathbb{R}$, draw a vertical line to the graph of function $f_a(x) = ax$. The vertical line intersects the graph of $f_a(x)$ at the point

$$z_1 = (x_0, f_a(x_0)).$$

Step II. From the point $(x_0, f_a(x_0))$, draw a horizontal line that intersects the diagonal line $y = x$ at the point

$$(f_a(x_0), f_a(x_0)) = z_2.$$

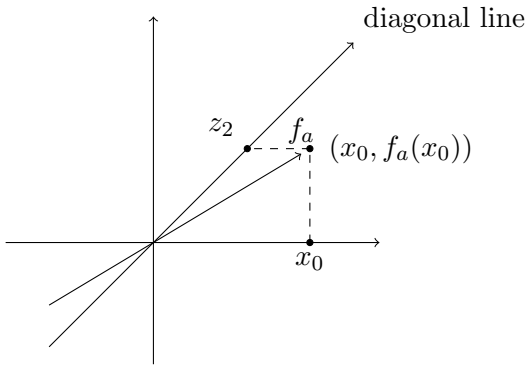
Step III. From point $z_2 = (f_a(x_0), f_a(x_0))$, draw a vertical line on the graph $f_a(x) = y$ that intersects at the point

$$z_3 = (f_a(x_0), f_a^2(x_0)).$$

Continue the process:

- vertically on the graph $y = f_a$,
- horizontally on the diagonal $y = x$,

we can visualize the orbit of a point x_0.

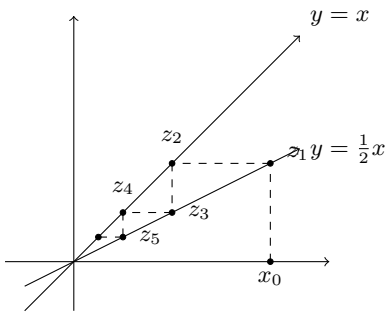


7. Example

Let $a = \dfrac{1}{2}$.

$$f_a(x) = \frac{1}{2}x.$$

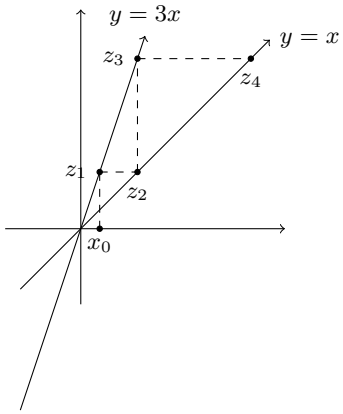
Slope is less than 1.



The diagonal line $y = x$ has slope $= 1 \Rightarrow \theta = 45°$.

So any line with slope < 1 has inclination angle between 0 and 45°. As the diagram shows, the staircase spirals **inward**, converging to the fixed point 0.

For $a = 3 \Rightarrow f_a(x) = 3x$ (so $|a| > 1$):



As the cobweb moves outward, the orbit is **moving to** $+\infty$ for $a = 3$ (since $|a| > 1$), spiraling away from the origin along the steeper line $y = 3x$ relative to the diagonal $y = x$.