Module 3

Lecture 3

Measurable Functions, Simple Functions & Integration of Non-Negative Functions

Throughout this lecture, $(X, \mathcal{A}, \mu)$ denotes a measure space.

1 Measurable Functions

Lemma 1.1

If $f, g \colon X \to \mathbb{R}$ are two real-valued measurable functions, then

$$\max(f, g) \qquad \text{and} \qquad \min(f, g)$$

are measurable.

Proof. Let $h = \max(f, g)$ and $k = \min(f, g)$, so $h, k \colon X \to \mathbb{R}$. Then for every $a \in \mathbb{R}$,

$$h^{-1}(a, +\infty] = f^{-1}(a, +\infty] \cup g^{-1}(a, +\infty],$$

and also

$$k^{-1}(a, +\infty] = f^{-1}(a, +\infty] \cap g^{-1}(a, +\infty].$$

Both right-hand sides are measurable, being a union and an intersection of measurable sets. □

1.1 Positive and Negative Parts

Let $f \colon X \to \overline{\mathbb{R}}$ be an extended real-valued function.

Definition 1.2: Positive and negative parts

The *positive part* of f is $f^+ \colon X \to [0, +\infty]$,

$$f^+(x) = \max(f(x), 0).$$

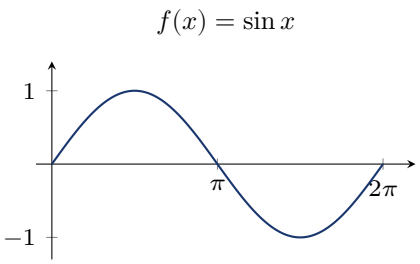
The *negative part* of f is $f^- \colon X \to [0, +\infty]$,

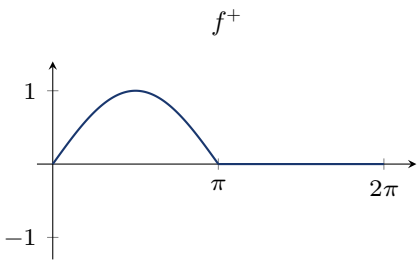
$$f^-(x) = \max(-f(x), 0).$$

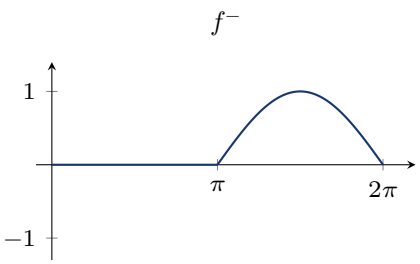
Clearly $f^+, f^- \geq 0$ are non-negative functions. Also,

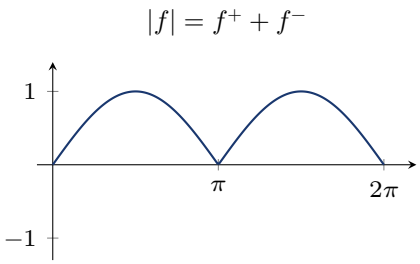
- $f = f^+ - f^-$ (the *decomposition* of f),
- $|f| = f^+ + f^-$.

Example 1.1. Let $f(x) = \sin x$ for $x \in [0, 2\pi]$. Then f^+ agrees with $\sin x$ on $[0, \pi]$ and vanishes on $[\pi, 2\pi]$; f^- vanishes on $[0, \pi]$ and equals $-\sin x$ on $[\pi, 2\pi]$; and $|f| = |\sin x|$ consists of two arches.









Lemma 1.3

f^+, f^- are measurable if and only if f is measurable. Moreover, if f is measurable then so is $|f|$.

1.2 Indicator Functions

Definition 1.4: Indicator function

Let $E \subseteq X$. The *indicator function* of E is $\mathbb{1}_E \colon X \to \mathbb{R}$ defined by

$$\mathbb{1}_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Lemma 1.5

$\mathbb{1}_E$ is measurable if and only if E is measurable.

Proof. For $c \in \mathbb{R}$ we have

$$\mathbb{1}_E^{-1}(c, +\infty] = \begin{cases} X, & c < 0, \\ E, & c \in [0, 1), \\ \emptyset, & c \geq 1. \end{cases}$$

Since $X, \emptyset \in \mathcal{A}$, every such preimage is measurable precisely when $E \in \mathcal{A}$. □

1.3 Simple Functions

Definition 1.6: Simple function

A real-valued function $f \colon X \to \mathbb{R}$ is called a *simple function* if the range of f is finite.

Example 1.2. The Dirichlet function $f \colon [0, 1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \cap [0, 1], \\ 1, & x \notin \mathbb{Q} \cap [0, 1], \end{cases}$$

is a simple function.

Canonical representation of a simple function. Let $f\colon X \to \mathbb{R}$ be a simple function with range

$$f(X) = \{c_1, c_2, \dots, c_n\}$$

(the c_i distinct), and let

$$E_i = f^{-1}(c_i), \qquad i = 1, \dots, n.$$

Clearly each E_i is measurable if f is measurable. Then

$$f(x) = \sum_{i=1}^{n} c_i \, \mathbb{1}_{E_i}(x)$$

is the *canonical representation* of f.

Remark.

- Simple functions are finite linear combinations of indicator functions $\mathbb{1}_{E_i}$, $i = 1, 2, \dots, n$.
- f is measurable if and only if each E_i is measurable, $i = 1, 2, \dots, n$.

Example 1.3. The Dirichlet function has canonical representation

$$f = 1 \cdot \mathbb{1}_{\mathbb{Q}^c \cap [0,1]} + 0 \cdot \mathbb{1}_{\mathbb{Q} \cap [0,1]}.$$

Theorem 1.7

Let $f, g\colon X \to \mathbb{R}$ be two simple functions. Then

- cf is simple for every $c \in \mathbb{R}$;
- $f \pm g$ is simple.

Remark. The set

$$S(X) = \{\, f\colon X \to \mathbb{R} \mid f \text{ is simple} \,\}$$

forms a linear space.

Note. We do not allow simple functions to assume the values $\pm\infty$.

Theorem 1.8: Simple Approximation Theorem

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f\colon X \to [0, +\infty]$ be a measurable function. Then there is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple functions on X such that

1. $0 \le f_1 \le f_2 \le \cdots \le f$;
2. $f_n \to f$ pointwise.

Moreover, $f_n \to f$ uniformly on any set $A \subseteq X$ on which f is bounded.

Exercises

Exercise 1.1. Let $f\colon X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable if and only if

(i) $f^{-1}(-\infty)$ is measurable,

(ii) $f^{-1}(+\infty)$ is measurable,

(iii) f is measurable on Y.

Exercise 1.2. If $f_n \colon X \to \overline{\mathbb{R}}$ is a sequence of measurable functions on X, then

$$\left\{ x \in X : \lim_{n\to\infty} f_n(x) \text{ exists} \right\}$$

is a measurable subset of X.

Exercise 1.3. If $f \colon X \to \overline{\mathbb{R}}$ and $f^{-1}(r, +\infty]$ is measurable for every $r \in \mathbb{Q}$, then f is measurable on X.

2 Integration of Non-Negative Functions

Let $(X, \mathcal{A}, \mu)$ be a measure space and set

$$L^+ = \{ f \colon X \to [0, +\infty] \mid f \text{ is measurable on } X \},$$

i.e. the collection of all non-negative extended real-valued measurable functions. We always adopt the convention $0 \cdot \infty = 0$.

2.1 Integral of an Indicator Function

$$\int_X \mathbb{1}_E \, d\mu = \int \mathbb{1}_E = \mu(E).$$

2.2 Integral of a Simple Function

Let $f \colon X \to [0, \infty)$ be a non-negative measurable simple function with canonical representation

$$f(x) = \sum_{i=1}^{n} c_i \, \mathbb{1}_{E_i}(x).$$

Then

$$\int_X f \, d\mu = \int f = \sum_{i=1}^{n} c_i \, \mu(E_i).$$

Notation.

$$\int_X f(x) \, d\mu(x) = \int_X f \, d\mu = \int f.$$

Integral over a subset. Let $A \in \mathcal{A}$ and let $\varphi = \sum_{i=1}^{n} c_i \mathbb{1}_{E_i}$ be a non-negative simple function. Then

$$\varphi \cdot \mathbb{1}_A = \sum_{i=1}^{n} c_i \, \mathbb{1}_{A \cap E_i},$$

and hence

$$\int_X \varphi \cdot \mathbb{1}_A = \sum_{i=1}^{n} c_i \, \mu(A \cap E_i) =: \int_A \varphi \, d\mu.$$

Proposition 2.1

Let φ, ψ be non-negative measurable simple functions. Then

(a) if $c \geq 0$, then $\int c\varphi = c \int \varphi$;

(b) $\int (\varphi + \psi) = \int \varphi + \int \psi$;

(c) $\varphi \leq \psi \implies \int \varphi \leq \int \psi$ (monotonicity);

(d) the map $A \mapsto \int_A \varphi \, d\mu$ is a measure on $\mathcal{A}$.

Proof. Exercise. □

2.3 Integral of a Function in L^+

Definition 2.2: Integral of a non-negative measurable function

For $f \in L^+$,

$$\int f \, d\mu = \sup\Big\{ \int \varphi \, d\mu \ : \ 0 \leq \varphi \leq f, \ \varphi \text{ simple} \Big\}.$$

Exercise 2.1. For $f, g \in L^+$:

- $f \leq g \implies \int f \leq \int g$;
- $\int cf = c \int f$ for $c \geq 0$.

Theorem 2.3: Monotone Convergence Theorem

If $\{f_n\}$ is a sequence in L^+ such that $f_j \leq f_{j+1}$ for all j, and

$$f = \lim_{n \to \infty} f_n,$$

then

$$\int f = \lim_{n \to \infty} \int f_n.$$

Monotonicity is essential here. Consider the following sequences on $\mathbb{R}$ with Lebesgue measure.

(1) $f_n = \mathbb{1}_{[n, n+1)}$. Then $f_n \to 0$ pointwise, but

$$\int f_n = 1 \ \text{ for all } n, \qquad \text{so} \qquad \lim_{n \to \infty} \int f_n = 1 \neq 0 = \int 0.$$

(2) $f_n = n \cdot \mathbb{1}_{(0, 1/n)}$. Then $f_n \to 0$ pointwise, but

$$\int f_n = 1 \ \text{ for all } n, \qquad \text{while} \qquad \int 0 = 0.$$

In both cases the sequence is not increasing, and the conclusion of the Monotone Convergence Theorem fails.