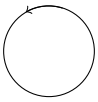
Circle Rotation

Circle Rotation

A circle can be represented in the following ways.



1. As a Subset of $\mathbb{R}^2$

All points in the plane $\mathbb{R}^2$ whose distance from the origin is 1:

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

In parametric form:

$$S^1 = \{(\cos 2\pi\theta, \sin 2\pi\theta) \mid 0 \leq \theta < 1\}.$$

2. In the Complex Plane $\mathbb{C}$

All $z \in \mathbb{C}$ whose modulus is equal to 1:

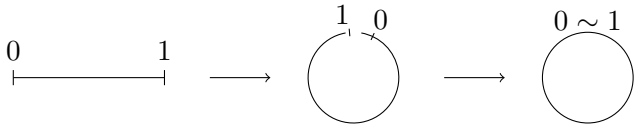
$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{2\pi i\theta} \mid 0 \leq \theta < 1\}.$$

3. Quotient Space

Line Segment:

$$S^1 = [0, 1]/\sim, \qquad 0 \sim 1.$$

S^1 forms as a quotient space by taking the closed segment $[0, 1]$ and gluing 0 and 1 together.



From the Real Line

$$S^1 = \mathbb{R}/\mathbb{Z} = \{x + \mathbb{Z} \mid x \in \mathbb{R}\}$$
$$x + \mathbb{Z} = y + \mathbb{Z} \iff x - y \in \mathbb{Z}.$$

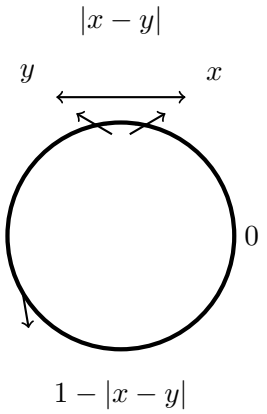
The quotient space $\mathbb{R}/\mathbb{Z}$ wraps the infinite line around a loop infinitely many times, with each interval $[n, n + 1]$ mapping onto the full circle S^1.

Metric on S^1

For $S^1 = [0,1]/\sim$, $0 \sim 1$, and $x, y \in S^1$, define

$$d(x,y) = \min\{\, |x-y|,\ 1-|x-y| \,\}$$

the minimum length of arc between x and y.



Exercise. *Show that d is a metric on S^1.*

Example. *Let $x = 0.2$, $y = 0.8$, with $x, y \in S^1$.*

$$\begin{aligned} |x-y| &= |-0.6| = 0.6, \\ 1-|x-y| &= 1-0.6 = 0.4, \\ d(x,y) &= \min\{0.6, 0.4\} = 0.4. \end{aligned}$$

Circle Rotation (Dynamics on S^1)

Let $\alpha \in S^1$. Define

$$f_\alpha : S^1 \to S^1$$

such that

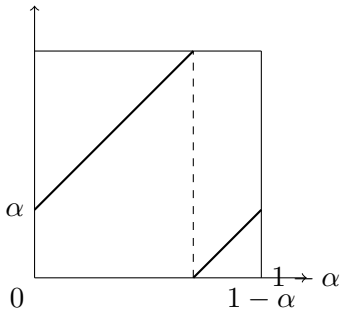
$$f_\alpha(x) = x + \alpha \mod 1.$$

Here, $x + \alpha \mod 1$ means only consider the fractional part of $x + \alpha$. For example,

$$1.25 \mod 1 = 0.25.$$

Also,

$$f_\alpha(x) = \begin{cases} x+\alpha, & x < 1-\alpha, \\ x+\alpha-1, & x \geq 1-\alpha. \end{cases}$$



f_α is Continuous

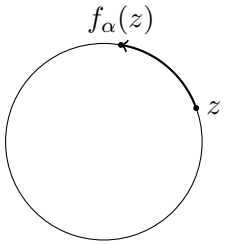
Indeed,

$$d(f_\alpha(x), f_\alpha(y)) = d(x, y) \qquad \forall\, x, y \in S^1,$$

so f_α is a distance-preserving (isometry) map. Therefore, f_α is continuous.

The map f_α is called **Circle Rotation** (sometimes **translation**) on S^1.

Remark. *For $z \in \mathbb{C}$, $ze^{i\theta}$ is the rotation of z with respect to angle θ.*



Let $z \in \mathbb{C}$ with $z \in S^1$, so

$$z = e^{2\pi i\theta}, \qquad \theta \in [0, 1).$$

Then

$$f_\alpha(z) = e^{2\pi i(\theta+\alpha)} = e^{2\pi i\alpha} \cdot e^{2\pi i\theta} = z\, e^{2\pi i\alpha}.$$

So $f_\alpha(z)$ is the rotation of z with respect to angle $2\pi\alpha$ (counter-clockwise).

Theorem (Dichotomy)

For $\alpha \in [0,1)$, let $f_\alpha : S^1 \to S^1$ be the circle rotation.

(a) If $\alpha \in \mathbb{Q}$, then for all $z \in S^1$, the orbit of z, $O^+(z)$, is finite.

i.e., all points $z \in S^1$ are periodic points. The period depends on α.

(b) If $\alpha \in \mathbb{Q}^c$ (α irrational), then for all $z \in S^1$, the forward orbit $O^+(z)$ is dense in S^1. i.e.,

$$\overline{O^+(z)} = S^1.$$

Proof

(a) Let $\alpha \in \mathbb{Q}$, the rational point.
Let $\alpha = \dfrac{p}{q}$, $p, q \in \mathbb{Z}$ and $q \neq 0$, $(p,q) = 1$.
Then let $z \in S^1$,

$$f_\alpha(z) = z + \alpha \mod 1 = z + \frac{p}{q} \mod 1.$$

$$f_\alpha^2(z) = f_\alpha(f_\alpha(z)) = f_\alpha\left(z + \frac{p}{q}\right) = z + \frac{p}{q} + \frac{p}{q} \mod 1 = z + \frac{2p}{q} \mod 1.$$

Continuing for q-times, we get

$$f_\alpha^q(z) = z + q \cdot \frac{p}{q} \mod 1.$$

$$f_\alpha^q(z) = z + p \mod 1 = z \qquad (\because\ p \in \mathbb{Z})$$

$$\Rightarrow\ f_\alpha^q(z) = z, \quad z \in S^1.$$

Therefore, every $z \in S^1$ is periodic with period q.

(b) Let $\alpha \in \mathbb{Q}^c$, α is irrational.
Claim: $O^+(z)$ is dense in S^1 $\ \forall z \in S^1$.

Dense set: A subset $D \subseteq X$, in a metric space X, is dense in X if for each $\varepsilon > 0$, for all $x \in X$, $\exists\, y \in D$ such that

$$d(x,y) < \varepsilon.$$

i.e., y is at ε-distance to x.

Our claim splits into two parts:

1. $O^+(z)$ is an infinite subset of S^1.
2. $O^+(z)$ is dense in S^1.

1. Infiniteness.

Let $O^+(z)$ be finite, then there exist $k, \ell \in \mathbb{N}$ $(k \neq \ell)$ such that

$$f_\alpha^k(z) = f_\alpha^\ell(z).$$

$$z + k\alpha \mod 1 = z + \ell\alpha \mod 1.$$

$$z + k\alpha + w = z + \ell\alpha \qquad \text{for some } w \in \mathbb{N}.$$

$$\Rightarrow\ k\alpha + w = \ell\alpha$$

$$\Rightarrow\ (k - \ell)\alpha + w = 0$$

$$\Rightarrow\ \alpha = \frac{w}{\ell - k}, \qquad \ell > k.$$

$$\Rightarrow\ \alpha \text{ is a rational point,}$$

which is a contradiction.

Therefore, $O^+(z)$ is an infinite subset of S^1.

2. Density.

Let $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \varepsilon.$$

Let $\{z, f_\alpha(z), f_\alpha^2(z), \ldots, f_\alpha^N(z)\} \subseteq O^+(z)$ be the finite subset of $O^+(z)$ of $N + 1$ points.

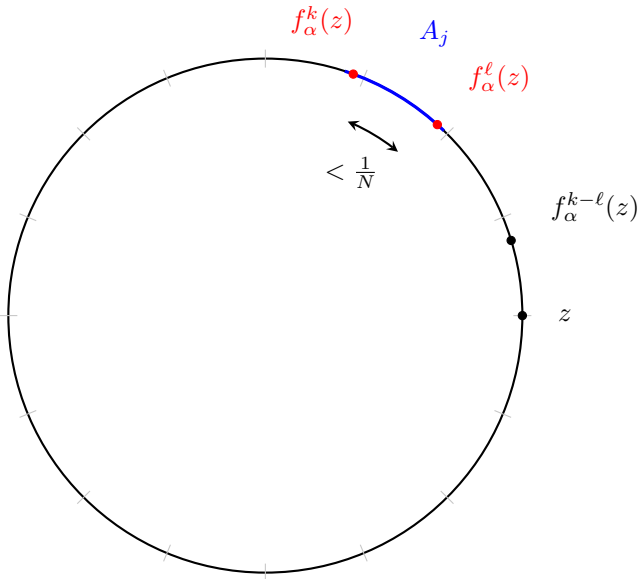
Make a partition of the circle S^1 into N equal parts, with length $\frac{1}{N}$. Let $(A_j)_{j=1}^N$ be the arcs of S^1, with $\ell(A_j) = \frac{1}{N}$ for $j = 1, 2, \ldots, N$.

We have N boxes (parts of the circle) and $N+1$ points, so by the Shoebox (pigeonhole) principle, at least one arc of S^1 has two points.

i.e., $\exists\, \ell, k \in \mathbb{N}$ such that

$$f_\alpha^k(z),\ f_\alpha^\ell(z) \in A_j \quad \text{for some } j \in \{1, 2, \ldots, N\},$$

with $\ell, k \leq N$, $\ell \neq k$.



i.e., $f_\alpha^k(z)$ and $f_\alpha^\ell(z)$ are ε-close:

$$d\big(f_\alpha^k(z), f_\alpha^\ell(z)\big) < \frac{1}{N} < \varepsilon.$$

$$\Rightarrow \; d\big(z, f_\alpha^{k-\ell}(z)\big) < \frac{1}{N}, \qquad (k > \ell)$$

since f_α is an isometry.

Also,

$$f_\alpha^{k-\ell}(z) = z + (k-\ell)\alpha \mod 1.$$

Let $\beta = (k-\ell)\alpha \mod 1$. Then

$$f_\alpha^{k-\ell}(z) = f_\beta(z).$$

The function $f_\beta : S^1 \to S^1$ is a rotation on S^1.

Also, for $z \in S^1$,

$$O_\beta^+(z) \text{ is the orbit of } z \text{ w.r.t. } \beta,$$

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Clearly,

$$O_\beta^+(z) \subseteq O_\alpha^+(z).$$

For any $w \in O_\beta^+(z)$, then

$$w = f_\beta^n(z) = f_\alpha^{(k-\ell)n}(z).$$

$$\Rightarrow \; w \in O_\alpha^+(z).$$

The β-rotation of the circle ensures the density. i.e., for any $x \in S^1$, $\exists\, m \in \mathbb{N}$ such that

$$d\big(x, f_\beta^m(z)\big) = d\big(x, f_\alpha^{m(k-\ell)}(z)\big) < \frac{1}{N} < \varepsilon.$$

$$\Rightarrow\ O^+(z) \text{ is dense in } S^1. \qquad \blacksquare$$