Backward Orbits, Pre-Images, and Backward Branches

1 Backward Orbit

Let $f : X \to X$ be bijective. Then f is invertible, the inverse map exists such that

$$f^{-1} : X \to X, \qquad f^{-1} \circ f = f \circ f^{-1} = \mathbb{I}_X \quad \text{(identity map)}.$$

We denote $f^{-n} = f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}$ (n times).

Intuitively, "backward" means going to the past. For example, if $x_0 \in X$, then $f(x_0)$ is the image of x_0 under f. Now

$$f^{-1}\big(f(x_0)\big) = f^{-1} \circ f(x_0) = \mathbb{I}(x_0) = x_0,$$

which recovers the initial condition.

Thus, if f is invertible, we can consider both forward and backward iterations.

Definition. *Let $f : X \to X$ be an invertible map. Let $x_0 \in X$ be the initial state. Then the forward and backward orbit of x_0 is defined as*

$$O^-(x_0) = \left\{ f^{-n}(x_0) \mid n \in \mathbb{Z}_{\leq 0} \right\} = \left\{ f^{-n}(x_0) \mid n \in \mathbb{Z}_{\geq 0} \right\}.$$

Also, the *full orbit* is

$$O(x_0) = O^+(x_0) \cup O^-(x_0) = \left\{ f^n(x_0) \mid n \in \mathbb{Z} \right\}.$$

That is, the orbit is a bi-infinite sequence.

2 Pre-Image

Let $f : X \to X$ be a dynamical system (not necessarily invertible).

Let $x_0 \in X$; then the *pre-image* of x_0 is the set

$$f^{-1}(x_0) = \left\{ x \in X \mid f(x) = x_0 \right\},$$

i.e., all points in X whose image is x_0.

For $n \geq 0$, the n-th pre-image of x_0 is

$$f^{-n}(x_0) = \left\{ x \in X \mid f^n(x) = x_0 \right\},$$

i.e., all points in X whose n-th iteration is x_0.

Example. *Let (S^1, f) be the doubling map. Then every point $x_0 \in S^1$ has exactly two pre-images (see Figure 1).*

$$\mathrm{Card}\left(f^{-1}(x_0)\right) = 2 \qquad \forall\, x_0 \in S^1.$$

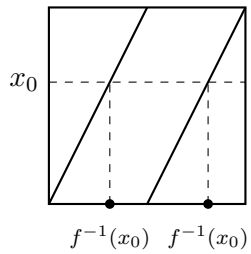


Figure 1: Graph of the doubling map: each horizontal line meets the graph twice.

Exercise. *Let $f : S^1 \to S^1$, $f(x) = nx$, $n \in \mathbb{N}$. Then*

$$\mathrm{Card}\left(f^{-1}(x_0)\right) = n \qquad \forall\, x_0 \in S^1.$$

Exercise (Exercise 2)**.** *Let $n = 5$, (S^1, f) be the doubling dynamical system. Find* $\mathrm{Card}\left(f^{-5}(x_0)\right)$. *In general, for $n \in \mathbb{N}$, find* $\mathrm{Card}\left(f^{-n}(x_0)\right) = ?$

3 Backward Branch

The *backward branch*, or a branch of pre-images, of x_0 is a sequence

$$\left\{f^{-n}(x_0)\right\}_{n=0}^{+\infty} \quad \text{such that} \quad f\big(f^{-n}(x_0)\big) = f^{-n+1}(x_0)$$

— a set if f is not invertible.

We denote

$$\hat{O}^-(x_0) = \bigcup_{n=0}^{+\infty} f^{-n}(x_0).$$

If f is invertible, then $\hat{O}^-(x_0) = O^-(x_0)$, i.e., the (unique) backward orbit of x_0.

Remark. *Sometimes the backward branch can be visualized by a tree diagram.*

Example. *Let (S^1, f) be the doubling map,*

$$f : S^1 \to S^1, \qquad f(x) = 2x \bmod 1 = \begin{cases} 2x, & x < 1/2 \\ 2x - 1, & x \geq 1/2. \end{cases}$$

For pre-images:

$$x < \tfrac{1}{2} \implies y = 2x \implies x = y/2, \quad x < \tfrac{1}{2},$$
$$x \geq \tfrac{1}{2} \implies y = 2x - 1 \implies x = \frac{y}{2} + \frac{1}{2}.$$

Hence

$$f^{-1}(x_0) = \left\{\frac{x_0}{2},\ \frac{x_0}{2} + \frac{1}{2}\right\} \qquad \textit{(two pre-images).}$$

In particular,

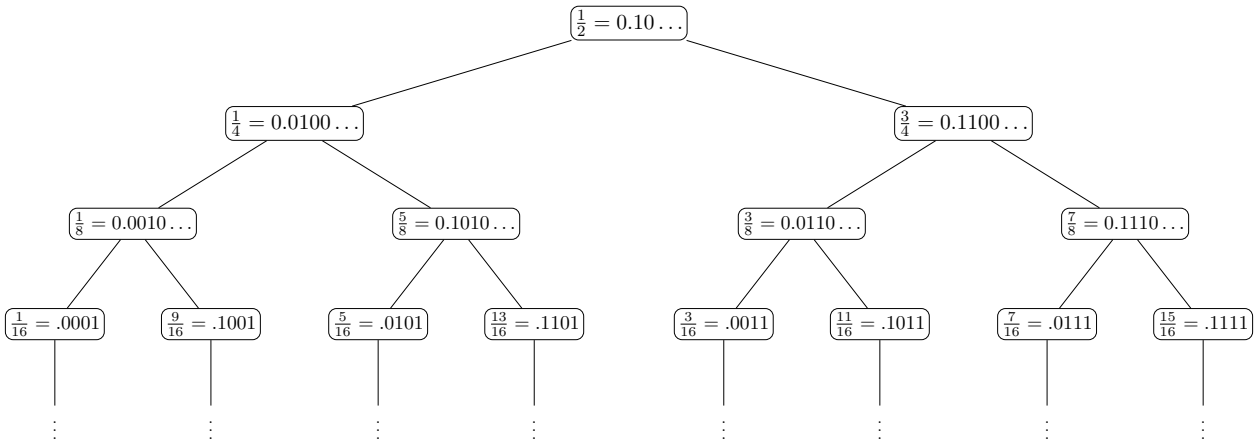
$$f^{-1}\big(\tfrac{1}{2}\big) = \big\{\tfrac{1}{4},\ \tfrac{1}{4}+\tfrac{1}{2}\big\} = \big\{\tfrac{1}{4},\ \tfrac{3}{4}\big\}.$$

$$f^{-2}\big(\tfrac{1}{2}\big) = \big\{f^{-1}(\tfrac{1}{4}),\ f^{-1}(\tfrac{3}{4})\big\} = \big\{\tfrac{1}{8},\ \tfrac{1}{8}+\tfrac{1}{2},\ \tfrac{3}{8},\ \tfrac{3}{8}+\tfrac{1}{2}\big\} = \big\{\tfrac{1}{2^3},\ \tfrac{5}{2^3},\ \tfrac{3}{2^3},\ \tfrac{7}{2^3}\big\}.$$

In general,

$$f^{-n}\big(\tfrac{1}{2}\big) = \Big\{\frac{p}{2^n} \;\Big|\; p \in \{1,3,5,7,9,\dots\}\Big\} \qquad \textit{(dyadic rational points).}$$

Tree structure of the backward branch



Invariant Sets

1 Invariant Sets

Let $A \subseteq X$.

1- A is called *forward invariant* if $f(A) \subseteq A$.

2- A is called *invariant* if $f(A) = A$.

3- A is called *fully-invariant* if $f^{-1}(A) = A$.

Clearly:

$$\text{fully invariant} \implies \text{invariant} \implies \text{forward invariant.}$$

But not conversely.

Example. *Let* $X = \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{x}{2}.$$

Let $A = [0, 1]$. *Then*

$$f(A) = [0, \tfrac{1}{2}] \subset [0, 1].$$

Therefore A *is forward invariant. But*

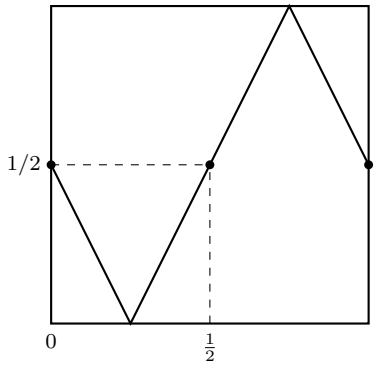
$$[0, \tfrac{1}{2}] = f(A) \neq A = [0, 1] \qquad \textit{not invariant.}$$

Also

$$f^{-1}(A) = [0, 2] \neq [0, 1] = A.$$

A *is not fully invariant.*

Example (Example 2)**.** *In the diagram below,* $X = [0, 1]$.

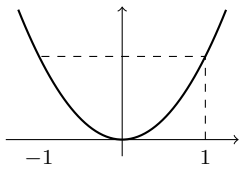


$$f^{-1}\big[0,\tfrac{1}{2}\big] = \big[0,\tfrac{1}{2}\big] \qquad f^{-1}\big[\tfrac{1}{2},1\big] = \big[\tfrac{1}{2},1\big].$$
$$\big[0,\tfrac{1}{2}\big] \textit{ is fully invariant.} \qquad \big[\tfrac{1}{2},1\big] \textit{ is fully invariant.}$$

Example (Example 3)**.** $f(x) = x^2$, $X = \mathbb{R}$.

$A = [0,1]$.

$$f(A) = A = [0,1] \qquad \textit{invariant.}$$
$$f^{-1}(A) = [-1,1] \supsetneq A \qquad \textit{not fully invariant.}$$



Exercise. *(a) Let (X,f) be a dynamical system. Let $x_0 \in X$ be a periodic point with period n. Show that the set $O^+(x_0)$ (forward orbit) is*

- *forward invariant.*
- *invariant.*
- *is $O^+(x_0)$ fully invariant?*

(b) If x_0 is not periodic, then show that the set $O^+(x_0)$ is forward invariant, but not fully invariant.

(c) Let $x_0 \in X$ be an aperiodic point in X. Let

$$A = O^+(x_0) \sqcup \hat{O}(x_0),$$

where

- *$O^+(x_0)$ is the forward orbit,*
- *$\hat{O}(x_0)$ is a backward branch of x_0.*

Show that A is forward invariant and invariant.

2 Smallest Fully Invariant Set Containing $O^+(x_0)$

Let $x_0 \in X$ be a fixed point in X. Then

$$O^+(x_0) = \{x_0\},$$

and

$$\hat{O}(x_0) = \big\{f^{-n}(x_0) \mid n \in \mathbb{N}\big\} \qquad \text{backward branch of } x_0.$$

The set

$$A = \{x_0\} \cup \hat{O}(x_0)$$

is

- forward invariant,
- invariant, and fully invariant.

Let x_0 not be a fixed point. Then the set

$$A = O^+(x_0) \cup \hat{O}(x_0)$$

is

- invariant, forward invariant, but not fully invariant.

Because if $y \in O^+(x_0)$, $y = x_n$, then

$$f^{-1}(y) \not\subseteq A.$$

In order to find the minimal fully invariant set that contains the orbit of x_0, the set

$$A = \bigcup_{y \in O^+(x_0)} \hat{O}(y)$$

is the smallest fully invariant set containing $O^+(x_0)$.

3 Exercises

1- If $O^+(x_0)$ is a finite set, then x_0 is periodic or pre-periodic. If f is invertible, then any pre-periodic orbit is periodic.

2- Show that if f is invertible then

$$f^{-1}(A) = A \iff f(A) = A.$$

3- Show that for any subset $A \subseteq X$, if $f^{-1}(A) = A$ then

$$f(A) = A.$$

4- Show that if $f^{-1}(A) = A$ then

$$f^{-1}(A^c) = A^c \qquad \text{where } A^c = X \setminus A.$$

5- Let $p \in X$ be a fixed point. Then what is the smallest fully invariant set A containing p, both in the invertible and non-invertible case?