DYNAMICAL SYSTEMS COURSE — BASIC PART

Module III

Lecture 1

Measure Theory: Motivation and σ-Algebras

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Synopsis

We motivate measure theory as an abstraction of length, area, and volume. Restricting to dimension 1, we axiomatize the classical notion of *length* on intervals, then ask whether it extends to a countably additive, translation-invariant set function on *all* of $\mathcal{P}(\mathbb{R})$. Using an irrational rotation of the circle (a first genuine dynamical system!) we show the answer is generally **no**. This forces us to restrict the domain of a measure to a well-behaved collection of sets: a σ-algebra. We develop the basic theory of algebras, σ-algebras, generated σ-algebras, and the Borel σ-algebra of a topological space.

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1 Measure Theory: Motivation

Measure is an extension of length, area, volume in space (an abstract space). Let us restrict our case to dimension 1 (for motivation). What is length?

1.1 Length as a set function

A length is an (extended real valued) *set function*, whose domain is the collection of all intervals; i.e., length is the size of an interval.

Let $\mathcal{I}(\mathbb{R})$ = collection of intervals of $\mathbb{R}$. Define

$$\ell : \mathcal{I}(\mathbb{R}) \longrightarrow \overline{\mathbb{R}}_{\geq 0}, \qquad \ell(J) = \begin{cases} b - a & \text{if } J \text{ bounded, with end points } a, b \ (a \leq b), \\ +\infty & \text{if } J \text{ unbounded.} \end{cases}$$

ℓ satisfies the following conditions:

(i) **Length is monotone**, i.e.,

$$J \subseteq K \implies \ell(J) \leq \ell(K).$$

(ii) **Length is translation invariant**, i.e.,

$$\ell(J + \alpha) = \ell(J) \qquad \forall\, \alpha \in \mathbb{R}.$$

(iii) **Length satisfies countable additivity**: for any pairwise disjoint sequence of intervals $J_1, J_2, \cdots \subseteq \mathcal{I}(\mathbb{R})$, i.e. $J_i \cap J_j = \varnothing$ for $i \neq j$,

$$\ell\left(\bigcup_{i \in \mathbb{N}} J_i\right) = \sum_{i \in \mathbb{N}} \ell(J_i).$$

1.2 Can length be extended to all of $\mathcal{P}(\mathbb{R})$?

Question 1.1

Is there a function $\mu : \mathcal{P}(\mathbb{R}) \to \overline{\mathbb{R}}_{\geq 0}$ (where $\mathcal{P}(\mathbb{R})$ is the power set of $\mathbb{R}$) which gives us the extension of length?

Answer

Generally, it is **not possible**.

We now justify this with a dynamical argument. Assume that

$$\nu : \mathcal{P}(\mathbb{R}) \longrightarrow \overline{\mathbb{R}}_{\geq 0}$$

is an extension of ℓ which satisfies properties (i)–(iii) above.

Let $S^1 = [0, 1)$ denote the unit circle, so that

$$\nu(S^1) = \ell\left([0, 1)\right) = 1.$$

For $\alpha \in \mathbb{Q}^c$ (i.e., α irrational), let

$$f : S^1 \to S^1, \qquad f(x) = x + \alpha \pmod 1$$

be the **irrational rotation** of the circle. Since α is irrational, for each $x_0 \in S^1$ the orbit $\mathcal{O}(x_0)$ is *dense* in S^1. For $x \in S^1$ we take the **full orbit**

$$\mathcal{O}(x) = \{f^i(x)\}_{i \in \mathbb{Z}},$$

which makes sense since f is an isometry (in particular a bijection of S^1).

1.3 A pathological, non-measurable set

Construct a set $S_0 \subseteq S^1$ such that S_0 contains exactly one point from each orbit in S^1 (such a choice exists by the Axiom of Choice). That is, for each $y \in S_0$, there exist $x \in S^1$ and $i \in \mathbb{Z}$ (corresponding to x) such that

$$y = f^i(x) \qquad \text{and} \qquad \mathcal{O}(x) \cap S_0 = \{y\}.$$

Clearly S_0 is an uncountable set, and $S_0 \subseteq S^1$.

Let $S_i := f^i(S_0)$ be the i-th translation of S_0. Then

$$\nu\big(f^i(S_0)\big) = \nu(S_0) \qquad \text{(by translation invariance).}$$

Claim

$S_i \cap S_j = \varnothing$ for all $i \neq j$, and $\bigcup_{i\in\mathbb{Z}} S_i = S^1$.

Proof of disjointness. Let $z \in S_i \cap S_j$ with $i \neq j$. Then $z \in S_i = f^i(S_0)$ and $z \in S_j = f^j(S_0)$, so there exist $x, y \in S_0$ with

$$z = f^i(x) \qquad \text{and} \qquad z = f^j(y),$$

hence $f^i(x) = f^j(y)$. Assume without loss of generality $i > j$; then

$$f^{\,i-j}(x) = y \in S_0.$$

This means $y \in \mathcal{O}(x)$, but already $x \in S_0$ — and from each orbit we chose exactly one point, a contradiction. Therefore $S_i \cap S_j = \varnothing$ for all $i \neq j$.

Also $\bigcup_{i\in\mathbb{Z}} S_i = S^1$, since $\{S_i\}_{i\in\mathbb{Z}}$ contains all orbits of S^1 (every point of S^1 lies on some orbit, and S_0 meets every orbit, so some translate S_i contains it). ■

1.4 The contradiction

By countable additivity of ν,

$$\sum_{i\in\mathbb{Z}} \nu(S_i) \;=\; \nu\Big(\bigcup_{i\in\mathbb{Z}} S_i\Big) \;=\; \nu(S^1) \;=\; 1,$$

and since $\nu(S_i) = \nu(S_0)$ for every i, this reads

$$\sum_{i\in\mathbb{Z}} \nu(S_0) = 1.$$

But now:

- if $\nu(S_0) = 0$, then $\sum_{i\in\mathbb{Z}} \nu(S_0) = 0 \neq 1$;
- if $\nu(S_0) > 0$, then $\sum_{i\in\mathbb{Z}} \nu(S_0) = +\infty \neq 1$.

Neither is possible — because of the pathological set S_0. Therefore, **normally we cannot extend length to the whole power set of** $\mathbb{R}$.

2 What We Need for an Extension

A well-defined domain.

2.1 σ-Algebra: the domain of a measure

Let $X \neq \varnothing$ and let $\mathcal{P}(X)$ denote the power set of X.

Definition 2.1 (Algebra)

An *algebra* $\mathcal{A} \subseteq \mathcal{P}(X)$ is a collection satisfying:

(i) $E \in \mathcal{A} \implies E^c \in \mathcal{A}$ (closed with respect to complement);

(ii) $\{E_1, E_2, \ldots, E_n\} \subseteq \mathcal{A} \implies \bigcup_{i=1}^{n} E_i \in \mathcal{A}$,

i.e., $\mathcal{A}$ is closed with respect to finite union.

Note

$E \in \mathcal{A} \implies E^c \in \mathcal{A} \implies E \cup E^c = X \in \mathcal{A}$, and $X^c = \varnothing \in \mathcal{A}$. By definition, an algebra always contains $\varnothing$ and the full set X.

Definition 2.2 (σ-algebra)

$\mathcal{A} \subseteq \mathcal{P}(X)$ is a *σ-algebra* if:

- $\mathcal{A}$ is an algebra;
- $\{E_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A} \implies \bigcup_{i\in\mathbb{N}} E_i \in \mathcal{A}$,

i.e., a σ-algebra is closed with respect to *countable* union.

Remark 2.1

For $\{E_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ a σ-algebra, we have $\{E_i^c\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$, and

$$\bigcap_{i\in\mathbb{N}} E_i = \Big(\bigcup_{i\in\mathbb{N}} E_i^c\Big)^c.$$

Therefore, a σ-algebra is also closed with respect to countable intersection.

2.2 Examples

- $\mathcal{A} = \{\varnothing, X\}$ and $\mathcal{A} = \mathcal{P}(X)$ are trivial examples of a σ-algebra. However, $\mathcal{P}(X)$ does not provide a *meaningful* σ-algebra if X is uncountable (cf. §1).
- For $E \subseteq X$: $\mathcal{A} = \{\varnothing, E, E^c, X\}$ is a σ-algebra.

Exercise 2.1 (co-countable σ-algebra)

Let X be uncountable. Then the collection

$$\mathcal{A} = \{E \subseteq X : E \text{ is countable, or } E^c \text{ is countable}\}$$

is a σ-algebra. $\mathcal{A}$ is known as the **co-countable σ-algebra**.

Definition 2.3 (Measurable space)

Let $X \neq \varnothing$ and let $\mathcal{A}$ be a σ-algebra on X. Then the pair

$$(X, \mathcal{A})$$

is called a **measurable space**.

3 Generated σ-Algebras and the Borel σ-Algebra

Question 3.1

Since we are in some dynamical system (X, f), a natural question: does a topological space form a measure space?

Note

A topology on X does *not* form a σ-algebra by itself (a topology need not be closed under complements or countable intersections of open sets).

Answer

Yes — via the notion of a *generated* σ-algebra.

Definition 3.1 (σ-algebra generated by a family of sets)

Let $\mathcal{E} \subseteq \mathcal{P}(X)$ be a family of subsets of X. Define

$$\sigma(\mathcal{E}) := \bigcap_{\alpha \in I} \{ \mathcal{A}_\alpha \mid \mathcal{A}_\alpha \text{ is a } \sigma\text{-algebra on } X,\ \mathcal{E} \subseteq \mathcal{A}_\alpha \}.$$

Naturally, at least one σ-algebra on X — namely $\mathcal{P}(X)$ — contains $\mathcal{E}$, so the above family of σ-algebras containing $\mathcal{E}$ is non-empty.

Exercise 3.1

$\sigma(\mathcal{E})$ forms a σ-algebra on X, and is the smallest σ-algebra on X containing $\mathcal{E}$. The σ-algebra $\sigma(\mathcal{E})$ is called **the σ-algebra generated by a family $\mathcal{E}$**.

Therefore, a σ-algebra generated by a topology on X is a natural σ-algebra on a topological space.

Definition 3.2 (Borel σ-algebra)

Let $(X, \mathcal{T})$ be a topological space. Then the σ-algebra generated by all open sets is called the **Borel σ-algebra**. The elements of this σ-algebra are called **Borel sets**.

3.1 What type of sets lie in a Borel σ-algebra?

- **G_δ-set**: a countable *intersection* of open sets.
- **F_σ-set**: a countable *union* of closed sets.
- **$G_{\delta\sigma}$-set**: a countable union of G_δ-sets.

- $F_{\sigma\delta}$**-set**: a countable intersection of F_σ-sets.

Note

δ and σ are used for *intersection* and *union*, respectively (from the German *Durchschnitt* = intersection, and *Summe* = union). All such sets are Borel, since $\mathcal{A}$ is closed under countable unions and countable intersections.

Exercise 3.2

Let $\mathcal{B}_{\mathbb{R}}$ be the Borel σ-algebra on $\mathbb{R}$. Then show that $\mathcal{B}_{\mathbb{R}}$ is generated by the open intervals in $\mathbb{R}$.

Note

In a measurable space $(X, \mathcal{A})$, any $E \in \mathcal{A}$ is called a **measurable set**. If $\mathcal{A}$ is the Borel σ-algebra, then E is called a **Borel measurable set**.

4 Measures

Definition 4.1 (Measure)

Let $(X, \mathcal{A})$ be a measurable space. An extended real-valued function

$$\mu : \mathcal{A} \longrightarrow \overline{\mathbb{R}}'_{\geq 0}, \qquad \overline{\mathbb{R}}'_{\geq 0} := [0, +\infty],$$

is called a **measure** if:

(a) $\mu(\varnothing) = 0$;

(b) for $\{E_i\}_{i\in\mathbb{N}} \subseteq \mathcal{A}$ such that $E_i \cap E_j = \varnothing$ for all $i \neq j$,

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} \mu(E_i) \qquad (\mu \text{ is } \textit{countably additive}).$$

Then μ is called a **measure on** X. The measurable space $(X, \mathcal{A})$ together with μ, written $(X, \mathcal{A}, \mu)$, is called a **measure space**.

Remark 4.1 (measures are finitely additive)

If μ is a measure on X, then μ is also *finitely additive*.

Proof. Let $\{E_i\}_{i=1}^{K} \subseteq \mathcal{A}$ be pairwise disjoint, i.e. $E_i \cap E_j = \varnothing$ for $i \neq j$. Extend this to a countable collection by setting $E_i := \varnothing$ for all $i \geq K$. Since $\mu(\varnothing) = 0$, countable additivity gives

$$\sum_{i=1}^{\infty} \mu(E_i) = \mu\Big(\bigcup_{i=1}^{\infty} E_i\Big).$$

But $E_i = \varnothing$ for all $i \geq K$ forces $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{K} E_i$ and $\sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{K} \mu(E_i)$, so

$$\mu\Big(\bigcup_{i=1}^{K} E_i\Big) = \sum_{i=1}^{K} \mu(E_i),$$

i.e., μ is finitely additive. ∎

Definition 4.2 (finite, probability, σ-finite)

Let $(X, \mathcal{A})$ be a measurable space and μ a measure on X.

(i) μ is **finite** if $\mu(X) < +\infty$.

(ii) μ is a **probability measure** if $\mu(X) = 1$.

(iii) μ is **σ-finite** if $X = \bigcup_{i=1}^{\infty} E_i$ with $E_i \in \mathcal{A}$ for all i, and $\mu(E_j) < +\infty$ for all j.

4.1 Examples

1. Counting measure. Let X be a countable set and $\mathcal{A} = \mathcal{P}(X)$. Define $\mu : \mathcal{A} \to \mathbb{R}_{\geq 0}$ by

$$\mu(A) = \begin{cases} |A| & A \text{ is finite,} \\ +\infty & A \text{ is infinite.} \end{cases}$$

Then μ is a measure on X, called the **counting measure**.

2. Dirac delta. Let $x_0 \in X$ and $\mathcal{A} = \mathcal{P}(X)$. Define

$$\delta_{x_0} : \mathcal{P}(X) \longrightarrow [0, +\infty], \qquad \delta_{x_0}(A) = \begin{cases} 1 & x_0 \in A, \\ 0 & x_0 \notin A. \end{cases}$$

δ_{x_0} is a measure, called the **Dirac delta** (at x_0).

Exercise 4.1 (a countable/co-countable measure)

Let X be an uncountable set and $\mathcal{A}$ the σ-algebra of countable-or-co-countable sets (§2.2). Define $\mu : \mathcal{A} \to \overline{\mathbb{R}}'_{\geq 0}$ by

$$\mu(E) = \begin{cases} 0 & E \text{ is countable,} \\ 1 & E \text{ is co-countable.} \end{cases}$$

Show that μ is a measure on X.

Exercise 4.2 (is this a measure?)

Let X be an infinite set and $\mathcal{A} = \mathcal{P}(X)$. Define $\mu : \mathcal{A} \to \overline{\mathbb{R}}'_{\geq 0}$ by

$$\mu(A) = \begin{cases} 0 & A \text{ is finite,} \\ +\infty & A \text{ is infinite.} \end{cases}$$

Is μ a measure on X? Justify your answer carefully using the definition of countable additivity.

5 Exercises

Exercise 5.1

Let $\mu_1, \mu_2, \ldots, \mu_n$ be measures on $(X, \mathcal{A})$ and let $a_1, a_2, \ldots, a_n \in [0, +\infty)$. Show that

$$\mu = \sum_{i=1}^{n} a_i \mu_i$$

is a measure on X.

Exercise 5.2

Let μ be a finite measure on X. Show that for every $E \in \mathcal{A}$, $\mu(E) < +\infty$.

Exercise 5.3

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $E \in \mathcal{A}$. Define

$$\mu_E : \mathcal{A} \longrightarrow \overline{\mathbb{R}}'_{\geq 0}, \qquad \mu_E(A) = \mu(A \cap E) \quad \text{for } A \in \mathcal{A}.$$

Show that μ_E is a measure on X.

Looking ahead

With $(X, \mathcal{A}, \mu)$ a full measure space in hand, the next lecture returns to the circle rotation (S^1, f) from §1: we construct Lebesgue measure on $\mathbb{R}$ (and on S^1) explicitly, verify it is f-invariant, and use invariance under f as the starting point for ergodic theory — the study of how measures transform under the dynamics.