DYNAMICAL SYSTEMS COURSE — BASIC PART

Module III

Lecture 2

Properties of Measures, Outer Measures, and Measurable Functions

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Synopsis

We record the basic calculus of a measure — monotonicity, subadditivity, and continuity from below and above — and see, via the counting measure, that continuity from above genuinely needs a finiteness hypothesis. We then answer the natural question left open in Lecture 1: *how do we actually build a well-behaved σ-algebra and a measure on it?* The tool is Carathéodory's outer measure construction, which starts from a length-type set function on an elementary family and manufactures, on the other side, a complete measure on a genuine σ-algebra — Lebesgue measure on $\mathbb{R}$ being the flagship example. We close with measurable functions: the class of maps between measurable spaces that measure theory (and, eventually, integration) can act on.

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1 Basic Properties of a Measure

Theorem 1.1

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) **Monotonicity.** If $E, F \in \mathcal{A}$ and $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

(b) **Subadditivity.** If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$, then

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \;\leq\; \sum_{i=1}^{\infty} \mu(E_i).$$

(c) **Continuity from below.** If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \;=\; \lim_{j \to +\infty} \mu(E_j) \;=\; \sup_{j} \mu(E_j).$$

(d) **Continuity from above.** If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_1) < +\infty$, then

$$\mu\Big(\bigcap_{i=1}^{\infty} E_i\Big) \;=\; \lim_{n \to +\infty} \mu(E_n) \;=\; \inf_{n} \mu(E_n).$$

Exercise 1.1 (proof of Theorem 1.1)

Prove parts (a)–(d) above. *Hint:* for (b)–(d), disjointify the sequence first — set $F_1 = E_1$ and $F_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1})$ for $n \geq 2$ in the increasing case, apply countable additivity to the F_n, and use monotonicity where needed. Part (d) follows from (c) applied to $E_1 \setminus E_n$, using $\mu(E_1) < +\infty$ to justify subtracting.

Remark 1.1 (finiteness in (d) is essential)

The hypothesis $\mu(E_1) < +\infty$ in part (d) cannot be dropped. Consider the counting measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, and let

$$E_j = \{\, n \in \mathbb{N} : n \geq j \,\}.$$

Then $E_1 \supseteq E_2 \supseteq \cdots$, and $\mu(E_j) = +\infty$ for every j (each E_j is infinite) — yet

$$\bigcap_{i=1}^{\infty} E_i = \varnothing \quad \Longrightarrow \quad \mu\Big(\bigcap_{i=1}^{\infty} E_i\Big) = \mu(\varnothing) = 0 \;\neq\; \lim_{j \to +\infty} \mu(E_j) = +\infty.$$

Here $\mu(E_1) = +\infty$, so the hypothesis of (d) fails, and indeed its conclusion fails too.

1.1 Null sets, almost everywhere, and completeness

Let $(X, \mathcal{A}, \mu)$ be a measure space.

Definition 1.1 (μ-null set)

A set $E \subseteq X$ is called **μ-null** if $\mu(E) = 0$.

Definition 1.2 (almost everywhere)

We say that a property P holds **almost everywhere** on X (a.e. on X) if

$$\mu\{\, x \in X : P \text{ fails to hold at } x \,\} = 0.$$

For example, if we say that a real-valued function $f : X \to \mathbb{R}$ is continuous a.e. on X, we mean that the set $\{x : f \text{ is not continuous at } x\}$ has zero measure.

Definition 1.3 (complete measure)

A measure whose domain includes *every* subset of a null set is called **complete**. That is, μ is complete if

$$\mu(A) = 0 \implies E \in \mathcal{A} \text{ for every } E \subseteq A.$$

2 Outer Measures and Carathéodory's Theorem

We now address the question left open by Lecture 1: how does one actually *find* a good σ-algebra and a good measure on a space X? The tool is to define an outer measure on all of $\mathcal{P}(X)$ first, and then cut it down.

Definition 2.1 (outer measure)

An **outer measure** $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_{\geq 0}$ is a set function satisfying:

(i) $\mu^*(\varnothing) = 0$;

(ii) $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$;

(iii) $\mu^*\big(\bigcup_i A_i\big) \leq \sum_i \mu^*(A_i)$ (μ^* is *subadditive*).

Question 2.1

How do we construct an outer measure on a set X?

Definition 2.2 (elementary family)

A family of sets $\Sigma \subseteq \mathcal{P}(X)$ is called an **elementary family** if the following properties hold on Σ:

(a) $\varnothing \in \Sigma$;

(b) if $E, F \in \Sigma$ then $E \cap F \in \Sigma$;

(c) if $E \in \Sigma$ then E^c is a finite disjoint union of members of Σ.

Example

The collection of all intervals of $\mathbb{R}$ forms an elementary family.

Proposition 2.1

Let $\Sigma \subseteq \mathcal{P}(X)$ be an elementary family, and let $\rho : \Sigma \to [0, +\infty]$ be such that $\varnothing \in \Sigma$, $X \in \Sigma$, and $\rho(\varnothing) = 0$. For any $A \subseteq X$, define

$$\mu^*(A) \;=\; \inf\Big\{ \sum_{i=1}^{\infty} \rho(E_i) \;:\; E_i \in \Sigma \text{ and } A \subseteq \bigcup_{i=1}^{\infty} E_i \Big\}.$$

Then μ^* is an outer measure.

Example (explanation)

In $\mathbb{R}$, for any $A \subseteq \mathbb{R}$ there is a countable collection of open intervals $\{I_i\}_{i=1}^{\infty}$ such that $A \subseteq \bigcup_i I_i$. Let $\rho =$ length of an interval. We define

$$\mu^* : \mathcal{P}(\mathbb{R}) \longrightarrow [0, +\infty], \qquad \mu^*(A) = \inf\Big\{ \sum_{i=1}^{\infty} \ell(I_i) \;\Big|\; A \subseteq \bigcup_{i} I_i \Big\},$$

where $\ell(I_i)$ is the length of the interval I_i. Then μ^* is called the **Lebesgue outer measure** on $\mathbb{R}$.

Definition 2.3 (Carathéodory measurability)

A set $A \subseteq X$ is called **μ^*-measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \forall\, E \subseteq X.$$

Proposition 2.2

The collection of all μ^*-measurable sets forms a σ-algebra on X:

$$\mathcal{A} = \{\, E \subseteq X : E \text{ is } \mu^*\text{-measurable} \,\}.$$

Example

In $\mathbb{R}$: $\mathcal{A} = \{\, E \subseteq \mathbb{R} : E \text{ is Lebesgue measurable} \,\}$ is a σ-algebra — the σ-algebra of all **Lebesgue measurable sets**, i.e., the extension of the elementary family of all intervals.

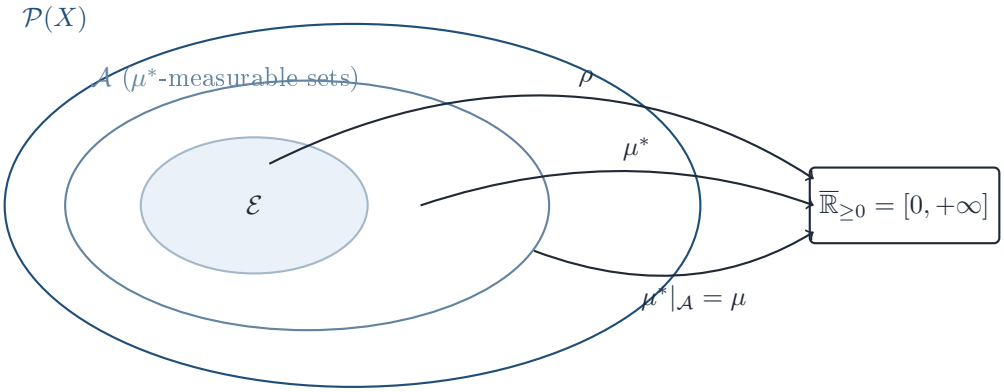
Theorem 2.3 (Carathéodory's Theorem)

If μ^* is an outer measure on X, the collection $\mathcal{A}$ of all μ^*-measurable sets forms a σ-algebra, and the restriction of μ^* to $\mathcal{A}$ is a **complete measure**.

Remark 2.1 (observation)

Lebesgue measure arises precisely as the measure furnished by Carathéodory's Theorem, applied to the Lebesgue outer measure above. In this sense, **Lebesgue measure is the length-type measure on $\mathbb{R}$.**

2.1 The Carathéodory construction, schematically



Reading the diagram: ρ is the elementary length-type function, defined only on the elementary family $\mathcal{E}$; μ^* extends it (via the infimum construction) to *all* of $\mathcal{P}(X)$, at the cost of losing countable additivity in general; restricting μ^* back down to the σ-algebra $\mathcal{A}$ of μ^*-measurable sets recovers a genuine, complete, countably additive measure $\mu = \mu^*|_{\mathcal{A}}$.

Remark 2.2 ($m(\mathbb{Q}) = 0$)

The Lebesgue measure of any countable set is zero. In particular, in $\mathbb{R}$, $m(\mathbb{Q}) = 0$. Therefore: *every real number is a.e. irrational*, or equivalently, *almost every real point is irrational.*

3 Measurable Functions

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. We now define what it means for a function $f : X \to Y$ to be measurable.

Definition 3.1 (pull-back image)

Let $f : X \to Y$ be a mapping. For $E \subseteq Y$,

$$f^{-1}(E) = \{\, x \in X : f(x) \in E \,\}$$

is called the **pull-back image** of E.

Fact

Let $\mathcal{B}$ be a σ-algebra on Y. Then

$$\mathcal{A} := f^{-1}(\mathcal{B}) = \left\{\, f^{-1}(E) : E \in \mathcal{B} \,\right\}$$

is a σ-algebra on X. Consequently $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are two measurable spaces.

Definition 3.2 ($(\mathcal{A}, \mathcal{B})$-measurable mapping)

A mapping $f : X \to Y$ is called $(\mathcal{A}, \mathcal{B})$-**measurable**, or just **measurable**, if

$$f^{-1}(E) \in \mathcal{A} \qquad \forall\, E \in \mathcal{B}.$$

Proposition 3.1 (composition)

If $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable and $g : Y \to Z$ is $(\mathcal{B}, \mathcal{C})$-measurable, then

$$g \circ f : X \to Z \quad \text{is } (\mathcal{A}, \mathcal{C})\text{-measurable.}$$

That is, the composition of two measurable mappings is measurable.

Proposition 3.2 (testing measurability on a generating family)

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be two measurable spaces. If $\mathcal{B} = \sigma(\Sigma)$, i.e. $\mathcal{B}$ is generated by a family Σ, then

$$f : X \to Y \text{ is } (\mathcal{A}, \mathcal{B})\text{-measurable} \iff f^{-1}(E) \in \mathcal{A} \quad \forall\, E \in \Sigma.$$

Corollary 3.3

Let X, Y be two metric (or topological) spaces. Then every continuous function is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable, since

$$\forall\, U \subseteq Y \text{ open} \implies f^{-1}(U) \subseteq X \text{ open.}$$

Note

A function $f : \mathbb{R} \to \mathbb{R}$ is **Lebesgue measurable** if and only if for each $E \subseteq \mathbb{R}$, E Lebesgue measurable $\implies$ $f^{-1}(E)$ is Lebesgue measurable.

Proposition 3.4

If $(X, \mathcal{M})$ is a measurable space, a real-valued function $f : X \to \mathbb{R}$ is measurable if and only if

$$f^{-1}\big((a, b)\big) \in \mathcal{M} \qquad \forall\, a, b \in \mathbb{R}.$$

4 Exercises: Preimages and Measurable Functions

These exercises are all about computing, or reasoning directly with, preimages $f^{-1}(E)$ — since that is exactly what Definition 3.2 and Proposition 3.4 ask you to check.

Exercise 4.1 (equivalent preimage tests for measurability)

Let $f : X \to \overline{\mathbb{R}} = [-\infty, +\infty]$. Show that the following are equivalent:

(i) $f^{-1}\big((a, +\infty]\big) \in \mathcal{A}$ for every $a \in \mathbb{R}$;

(ii) $f^{-1}\big([a, +\infty]\big) \in \mathcal{A}$ for every $a \in \mathbb{R}$;

(iii) $f^{-1}\big([-\infty, a)\big) \in \mathcal{A}$ for every $a \in \mathbb{R}$;

(iv) $f^{-1}\big([-\infty, a]\big) \in \mathcal{A}$ for every $a \in \mathbb{R}$.

Hint: pair up (i)↔(iv) and (ii)↔(iii) using complements, then link (i) and (ii) using $[a, +\infty] = \bigcap_n (a - \frac{1}{n}, +\infty]$.

Exercise 4.2 (indicator functions)

Let $A \subseteq X$ and let $\mathbb{1}_A : X \to \mathbb{R}$ be its indicator function. Show that $\mathbb{1}_A$ is $\mathcal{A}$-measurable if and only if $A \in \mathcal{A}$. *Hint:* compute $\mathbb{1}_A^{-1}(E)$ explicitly for $E \subseteq \mathbb{R}$, distinguishing the four cases according to whether $0 \in E$ and whether $1 \in E$.

Exercise 4.3 (the Dirichlet function is measurable)

Let $f = \mathbb{1}_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ be the **Dirichlet function**,

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Show directly, by computing $f^{-1}\big((a,b)\big)$ for arbitrary $a < b$ in $\mathbb{R}$ and checking each case against the Lebesgue σ-algebra, that f is Lebesgue measurable. *Hint:* there are exactly four cases, according to whether (a,b) contains 0, contains 1, contains both, or contains neither — in every case the preimage is either $\varnothing$, $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^c$, and each of these is Lebesgue measurable ($\mathbb{Q}$ is countable, hence Lebesgue-null, hence measurable by completeness).

Exercise 4.4 (measurable, but nowhere continuous)

Show that the Dirichlet function $f = \mathbb{1}_{\mathbb{Q}}$ from the previous exercise is discontinuous at *every* point of $\mathbb{R}$. *Hint:* use the density of both $\mathbb{Q}$ and $\mathbb{Q}^c$ in $\mathbb{R}$. Conclude that Corollary 3.3 (continuous $\implies$ measurable) cannot be used to establish the measurability you just proved directly — measurability is a strictly weaker, and more flexible, requirement than continuity.

Exercise 4.5 (preimages under a piecewise function)

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & x < 0, \\ x & 0 \le x < 1, \\ 2 & x \ge 1. \end{cases}$$

Compute $f^{-1}\big((0,1)\big)$ and $f^{-1}\big([1,3)\big)$ explicitly as subsets of $\mathbb{R}$, and use Proposition 3.4 together with your computation (and the fact that f is continuous on each piece) to verify that f is Borel measurable.

Exercise 4.6 (sums, via preimages)

Let $f, g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. Verify the set identity

$$\{\, x \in X : f(x) + g(x) < a \,\} \;=\; \bigcup_{r \in \mathbb{Q}} \Big(\{x : f(x) < r\} \cap \{x : g(x) < a - r\} \Big),$$

and use it, together with the fact that $\mathbb{Q}$ is countable, to conclude that $f+g$ is $\mathcal{A}$-measurable.

Exercise 4.7 (max and min, via preimages)

Let $f, g : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. Express $\{x : \max(f,g)(x) > a\}$ and $\{x : \min(f,g)(x) > a\}$ each as a union or intersection of preimages of f and g, and conclude that $\max(f,g)$ and $\min(f,g)$ are both $\mathcal{A}$-measurable.

Exercise 4.8 (composing with a continuous function)

Let $(X, \mathcal{A})$ be a measurable space, let $f : X \to \mathbb{R}$ be measurable, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\varphi \circ f : X \to \mathbb{R}$ is measurable. *Hint:* combine Corollary 3.3 with Proposition 3.1. Use this to give a one-line proof that f^2 and $|f|$ are measurable whenever f is.